

### 3. AMPERE'S LAW

#### Who was Amperè?

André Marie Amperè (1775-1836) grew up in the S. of France but still managed to submit his first mathematical paper to the Academy de Lyon at age 13! The family was beset by tragedy: Amperè's sister died when he was only 17. Amperè's father was sent to the guillotine for upsetting the authorities in Paris. Ampere married Julie who is recorded to have said of him on their first meeting,

*"He has no manners; he is awkward, shy and presents himself badly"*

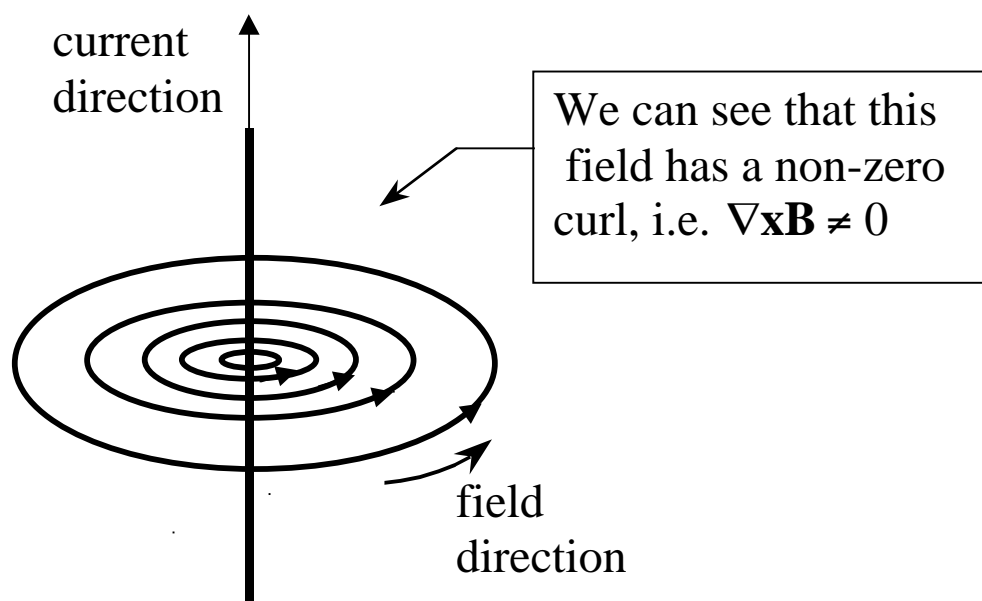
Despite all the setbacks Ampere made fundamental contributions to the establishment of EM theory in the 19<sup>th</sup> C and he is remembered in the name of the unit for electric current.

#### What does Amperè's law do?

Amperè's law gives us an elegant method for calculating the magnetic field – but only in cases where the symmetry of the problem permits (otherwise we must use the Biot-Savart Law). Amperè's law is to magnetics as Gauss' law is to electrostatics.

#### 3.1 The curl of $\mathbf{B}$ : $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$

Consider the magnetic field around a straight, current-carrying wire:



Previously we found (Section 2.2.2(3)) from the Biot-Savart law:

$$\mathbf{B} = \frac{\mu_0 I}{2\pi s} \hat{\phi} \quad \text{field due to long straight wire}$$

$\nearrow$   
 unit vector in the  
 azimuthal direction  
 (cylindrical coordinates)

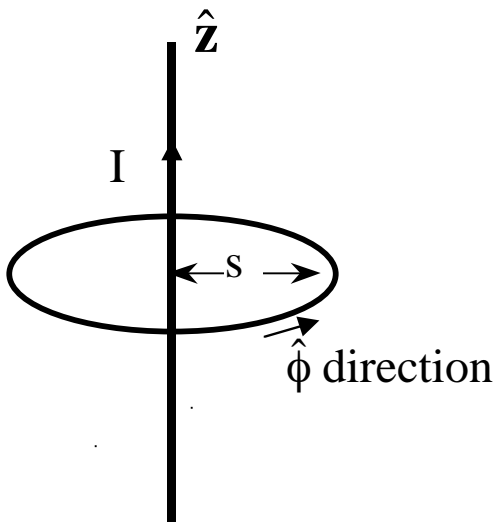
In cylindrical co-ordinates  $d\mathbf{l} = s \hat{s} + d\phi \hat{\phi} + dz \hat{z}$ ,  
 and with  $I$  along the  $z$ -axis,

$$\oint \mathbf{B} \cdot d\mathbf{l} = \frac{\mu_0 I}{2\pi} \oint \frac{1}{s} s d\phi$$

$$= \frac{\mu_0 I}{2\pi} \int_0^{2\pi} d\phi = \mu_0 I$$

$$= \mu_0 I$$

$\nwarrow$  integration encircles  
wire once



$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I$$

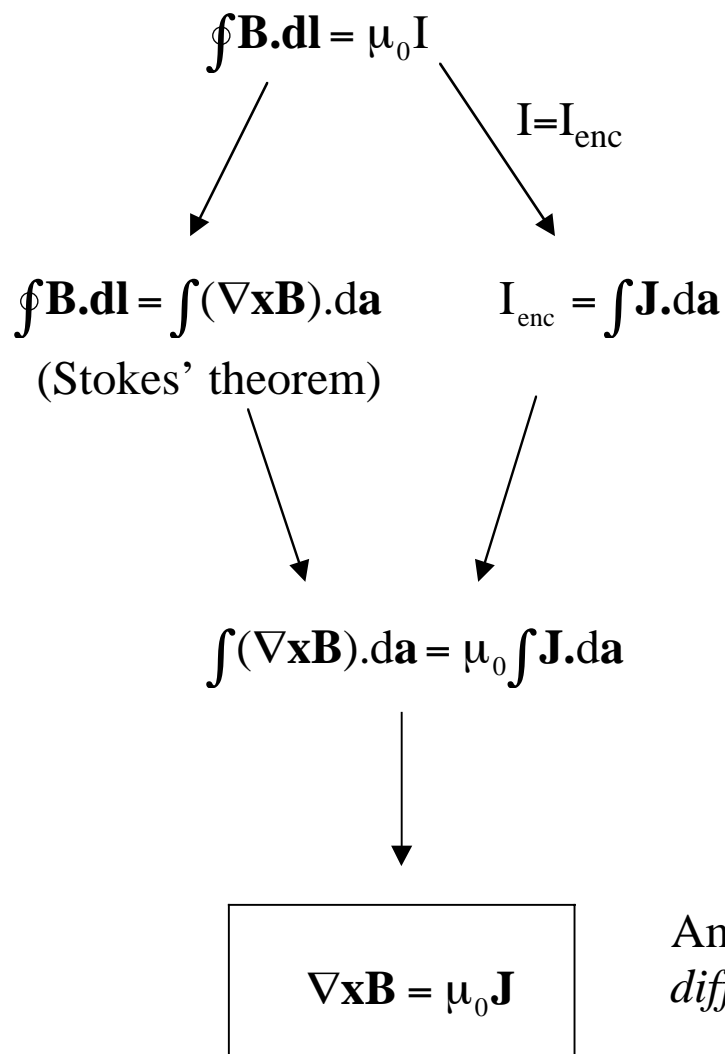
Amperè's law in  
*integral* form

$\uparrow$   
 Current enclosed ( $I = I_{\text{enc}}$ )  
 by the **Amperian loop**

If we have a volume current density enclosed (not just a *wire* carrying current  $I$ ) we use

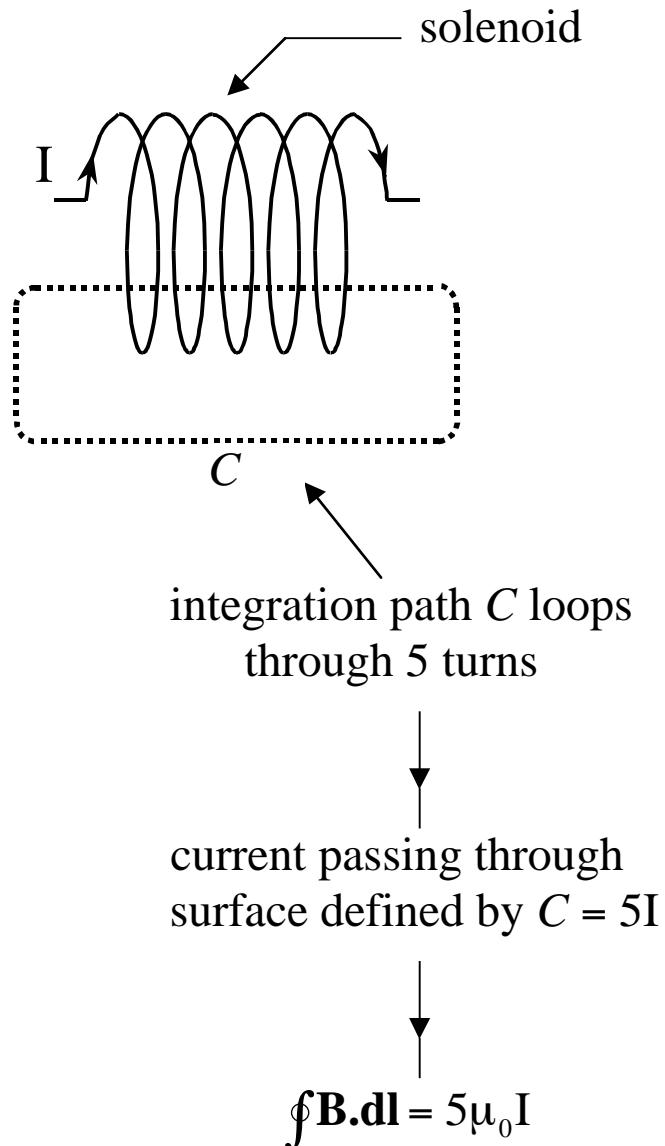
$$I_{\text{enc}} = \int \mathbf{J} \cdot d\mathbf{a}$$

and write Ampère's law



Ampère's law in  
*differential* form

We can arrange the closed path of integration to enclose multiple conductors (wires):



### 3.2 Applications of Ampere's law

Ampere's law is useful when there is high symmetry in the arrangement of the conductors. Standard examples are:

### 3.2.1 B due to a long straight wire:

$$B = \frac{\mu_0 I}{2\pi s}$$

$s$  = radial distance from wire

(see p226, Example 5.7, Griffiths)

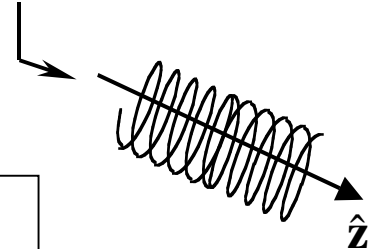
### 3.2.2 B due to a *long* solenoid

inside solenoid:  $B = \mu_0 n I \hat{z}$

outside solenoid:  $B = 0$

( $\hat{z}$  is along coil axis)

$n$  is number  
of turns per  
unit length

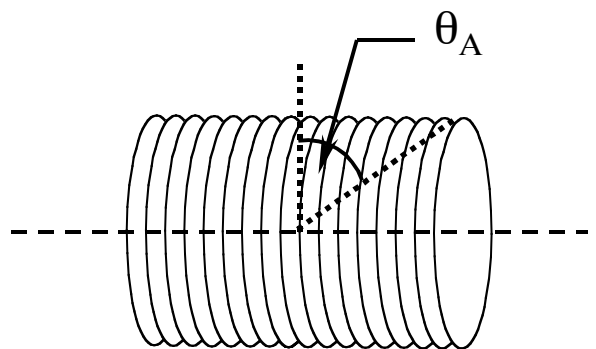


(see p227, Example 5.9, Griffiths)

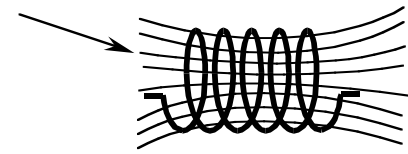
### 3.2.3 B for a *short* solenoid

At coil centre  $B = \mu_0 n I \sin \theta_A$

$\theta_A$  parameterises the  
*aspect ratio* of the coil

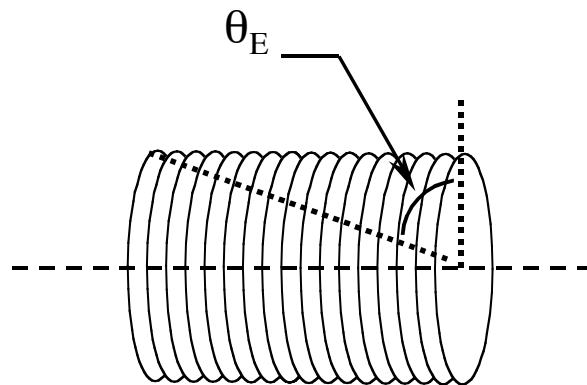


The field at the ends of a short solenoid is less than  $B$  at the centre because the lines of  $B$  splay out at the ends

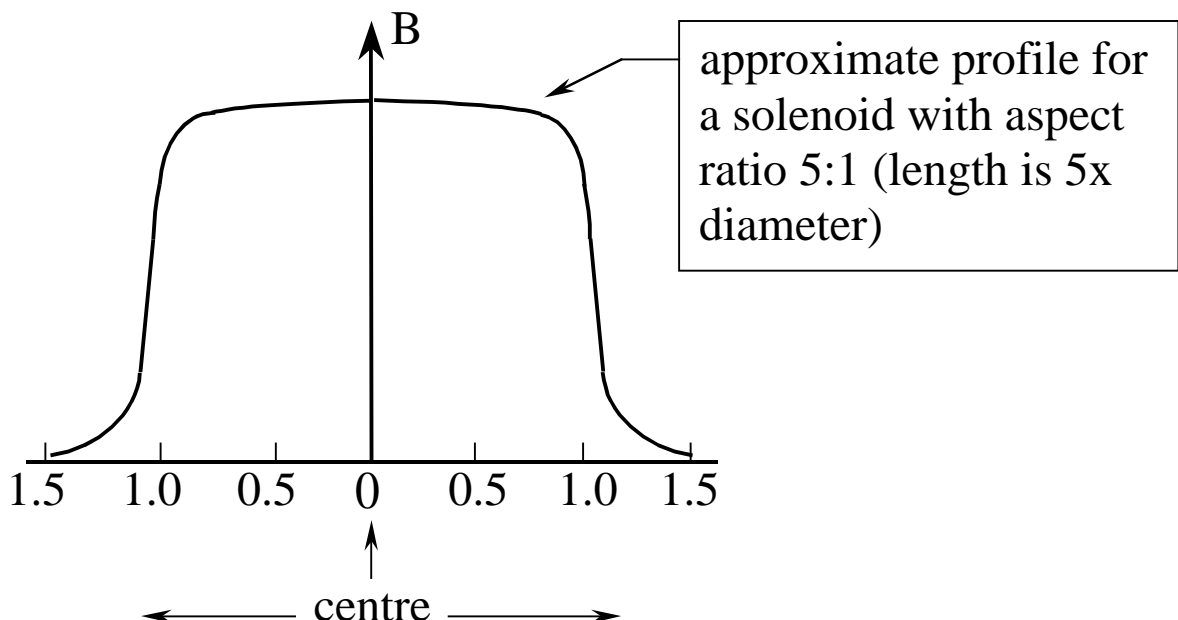


At the end of the short solenoid, on axis, we find<sup>†</sup>

$$B = \mu_0 n I \frac{\sin \theta_E}{2}$$



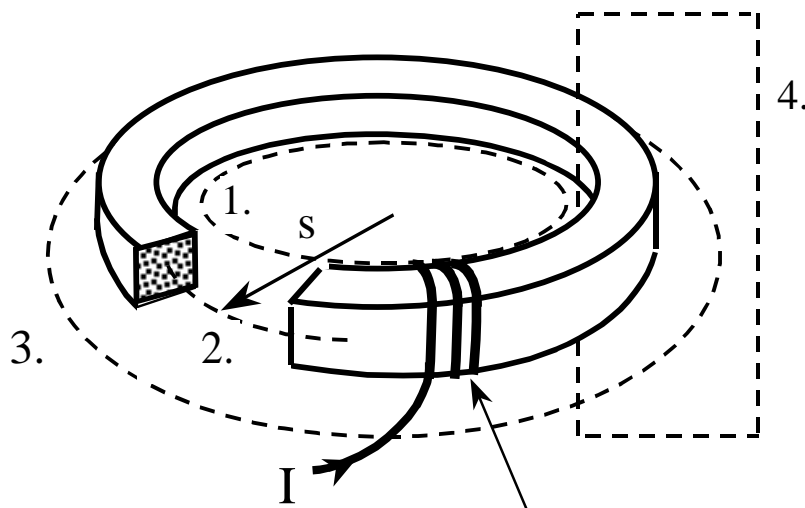
Along the solenoid axis the  $B$  field profile is, graphically :



† e.g. see p229 *Electromagnetism Principles and Applications*, Paul Lorrain and Dale Corson, pub. W.H. Freeman and Co., ISBN 0-7167-0064-6.

or, P317 *Electromagnetic Fields and Waves*, Paul Lorrain and Dale Corson, pub. W.H. Freeman & Co., ISBN 0-7167-0331-9.

**3.2.4** A more interesting example is the **toroid** - a 'doughnut' shape usually (but not necessarily) with a square or circular cross section (see Fig 5.38, p229, Griffiths).



current-carrying windings are close wound and continue around the torus, total  $N$  turns (only 3 turns are shown)

Choice of integration path 1. – 4.  $\Rightarrow$  we can find B-field inside torus, outside torus etc. Along path 2. (inside the torus' core)

Ampère's law gives

$$2\pi s \cdot B = \mu_0 NI$$

s is radius of toroid

N is number of turns on toroid

$$B = \frac{\mu_0 NI}{2\pi s} \quad (B \text{ in core of toroid})$$

- Outside the coil (path 3)  $B=0$  – useful for engineers! (Have a look at these sites to see how useful coils, toroidal transformers etc are.:

<http://www.toroid.com/rectifier0.htm>

<http://www.audiovideo101.com/dictionary/toroidal-transformer.asp>

<http://www.kinword.com.tw/>

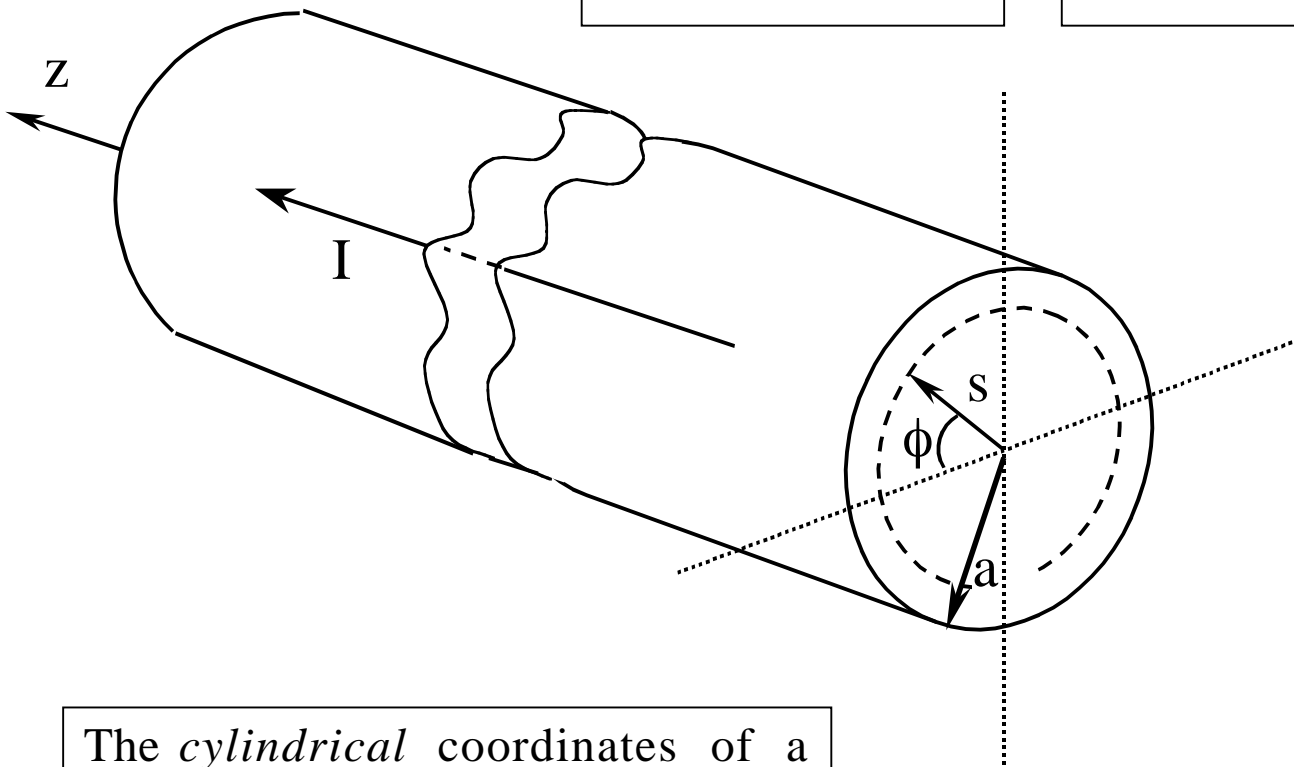
(see page 899 Tipler; page 230 of Griffiths for full treatment of the toroid)



### 3.2.5 Long cylindrical conductor

Conductor radius is  $a$ . The current  $I$  is in the  $z$ -direction

The current density is  $J = I/\pi a^2$



The *cylindrical* coordinates of a point in/around the conductor are  $s, \phi, z$  in the  $\hat{s} \hat{\phi} \hat{z}$  directions respectively (see Griffiths p43-45 if you can't remember!)

**Outside the conductor**, i.e. for  $s > a$ ,  $B$  is in the azimuthal direction and is independent of  $\phi$  ( $B$  has cylindrical symmetry).

Ampère's law for  $s > a$  gives

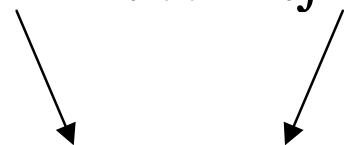
$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}$$

$$B(2\pi s) = \mu_0 I_{\text{enc}} = \mu_0 I$$

$$B = \frac{\mu_0 I}{2\pi s} \hat{\phi} \quad (\text{in the } \hat{\phi} \text{ direction})$$


**Inside the conductor**, at radius  $s$ , we use the current density version of Ampere's law with  $J = I/\pi s^2$  so

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}} = \mu_0 \int \mathbf{J} \cdot d\mathbf{a}$$



$$B(2\pi s) = \mu_0 J(\pi s^2)$$

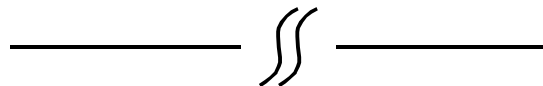
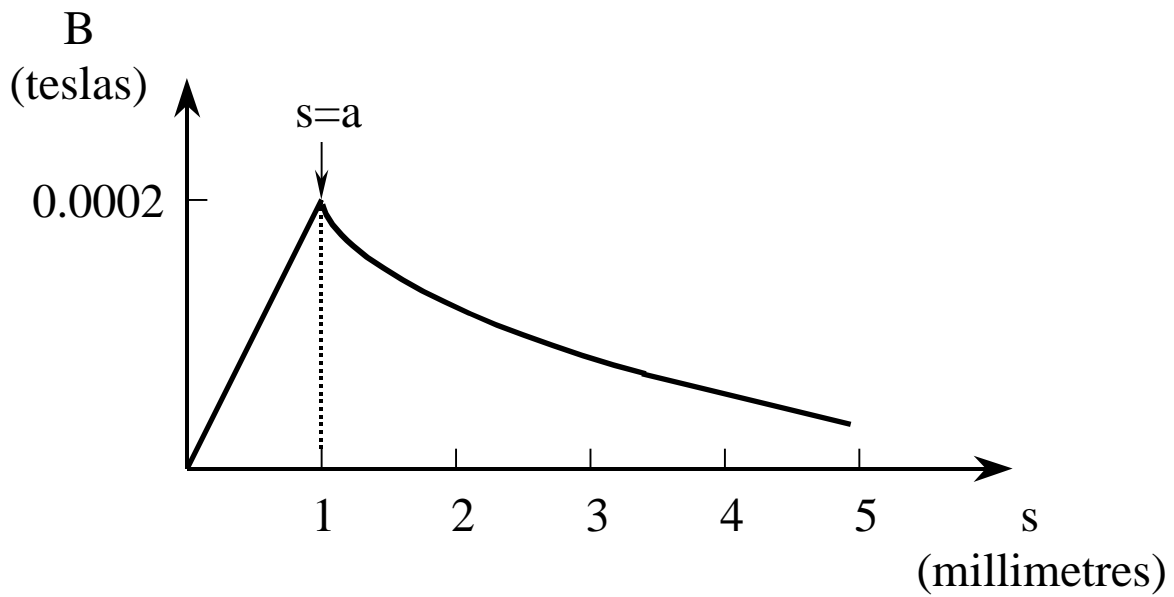
$$B = \frac{\mu_0 J s}{2} = \frac{\mu_0 I s}{2\pi a^2}$$


 $J = I/\pi s^2$

$$B = \frac{\mu_0 I s}{2\pi a^2}$$

inside the wire

Graphically, for a 1 mm radius wire carrying 1 amp:



### 3.4 The divergence of $\mathbf{B}$ : $\nabla \cdot \mathbf{B} = 0$

Taking the Biot-Savart law:

$$\mathbf{B} = \frac{\mu_0}{4\pi} I \oint \frac{d\mathbf{l} \times \hat{\mathbf{r}}}{r^2}$$

in the form where we have a volume current  $\mathbf{J}$  we have:

$$\mathbf{B} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J} \times \hat{\mathbf{r}}}{r^2} \quad (\text{p219 of Griffiths})$$

Now, taking the divergence of this ( $\nabla \cdot$ ),

$$\nabla \cdot \mathbf{B} = \frac{\mu_0}{4\pi} \int \nabla \cdot \left( \frac{\mathbf{J} \mathbf{r}}{r^2} \right) d\tau$$

Using the product rule

$$\nabla \cdot (\mathbf{A} \mathbf{B}) = \mathbf{B} \cdot (\nabla \mathbf{A}) + \mathbf{A} \cdot (\nabla \mathbf{B})$$

(this is Product rule #6 inside front cover of Griffiths)

we can write

$$\nabla \cdot \left( \frac{\mathbf{J} \mathbf{r}}{r^2} \right) = \underbrace{\frac{\mathbf{r}}{r^2} \cdot (\nabla \mathbf{J})}_{\text{this term} = 0} - \underbrace{\mathbf{J} \cdot \left( \nabla \frac{\mathbf{r}}{r^2} \right)}_{\text{this term} = 0}$$

this term = 0 because  
 $\nabla \mathbf{J} = 0$ .

$\nabla$  operates on field points at  $(x, y, z)$ . Current density  $\mathbf{J}$  is *not* a function of  $(x, y, z)$  but *is* a function of coordinates  $(x', y', z')$ . To see this, imagine a volume of current  $d\tau' = dx' dy' dz'$ ; the current density  $\mathbf{J}$  varies

this term = 0 because  
 $\left( \nabla \frac{\mathbf{r}}{r^2} \right) = 0$

The vector algebra result  $\nabla \cdot (\mathbf{r}^n \hat{\mathbf{r}}) = 0$  is well-known – at least by those who know it!

Thus,

$$\nabla \cdot \mathbf{B} = 0$$

General result: divergence of  $\mathbf{B}=0$  always

This is one of the two Maxwell equations of magnetostatics (Ampere's law is the other one).

If we use the divergence theorem  $\int \nabla \cdot \mathbf{d}\tau = \oint \mathbf{A} \cdot \mathbf{da}$  we have the **equivalent** statement

$$\oint \mathbf{B} \cdot \mathbf{da} = 0$$

The net magnetic flux out through any closed surface is zero

This tells us there are no magnetic 'monopoles' (perhaps we might say, no 'magnetic charges') in nature.

{ English physicist Paul Dirac predicted the existence of an 'anti-electron' or positron (which was discovered in 1932) following his solution of the relativistic Schrödinger equation. Dirac also predicted the existence of magnetic monopoles but these have not been found... }

If magnetic monopoles existed we would have an equation like  
Gauss' law for magnetostatics:

✓

$$\nabla \cdot \mathbf{E} = \frac{\rho_f + \rho_b}{\epsilon_0}$$

↑

Gauss' law in  
electrostatics

✗

$$\nabla \cdot \mathbf{B} \propto (\text{magnetic charge density})$$

↑

div  $\mathbf{B} \neq 0$  – this is never seen!

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