2. CURRENTS AND THE BIOT-SAVART LAW

2.1 Electric currents

An electric current is a movement of charge along a line (a wire), across a surface (a conducting sheet) or in a volume. We recall that

\[ I = \frac{dq}{dt} \quad \text{and} \quad I = nAv \]

Current is in amperes (A) or coulombs per second (Cs\(^{-1}\))

2.1.1 Line of charge

A current in a wire can be considered a line of charge of linear charge density \( \lambda \) moving at \( v \) ms\(^{-1}\).

If charge \( \lambda v \delta t \) passes point in time \( \delta t \), then...
I = \lambda v

velocity

Note: we have identified current as a vector (see p209 of Griffiths).

2.1.2 Force on current-carrying wire

For a steady current and fixed magnetic field,

\[ F_m = \int (vxB)\,dq \]

and

\[ dq = \lambda \,dl \]

so that

\[ F_m = \int (vxB)\lambda \,dl \]

\[ = \int (IxB)\,dl \quad (I \text{ and } v \text{ are in same dir’n}) \]

\[ = \int I(dlxB) \quad (I \text{ and } dl \text{ are in same dir’n}) \]
For current that is constant in magnitude,
\[ F_m = I \int (d \mathbf{l} \mathbf{xB}) \]

### 2.1.3 Surface and volume currents

When a surface charge density \( \sigma \) moves with velocity \( v \) over a surface we have a **surface current** and **surface current density** \( K \):

\[ K = \sigma v \]

and

\[ F_m = \int (v \mathbf{x} \mathbf{B}) \sigma \, d a = \int (K \mathbf{x} \mathbf{B}) \, d a \]

Similarly, when a current flows in some volume, we consider the **volume charge density** \( \rho \) and **volume current density** \( J \) and

\[ J = \rho v \]

and the magnetic force is

\[ F_m = \int (v \mathbf{x} \mathbf{B}) \rho \, d \tau = \int (J \mathbf{x} \mathbf{B}) \, d \tau \]

\( d \tau \) is a volume element
2.1.4 Equation of continuity

Using

\[ J = \frac{dI}{da_{\perp}} \]

we can write

\[ I = \int_S J da_{\perp} = \int_S J \cdot da \]

Using the divergence theorem (see the inside cover of Griffiths) we see that the total charge leaving volume \( V \) per unit time is

\[ \oint_S J \cdot da = \int_V (\nabla \cdot J) d\tau \]

\( da_{\perp} \) is an infinitesimal cross-section perpendicular to the current flow

\( J \) is the current per unit area perpendicular to the flow
(see Griffith, p32 for a discussion of the geometrical interpretation of the divergence theorem)

Now

\[ \int_V (\nabla \cdot \mathbf{J}) \, d\tau = -\frac{d}{dt} \int_V \rho \, d\tau = -\int_V \frac{\partial \rho}{\partial t} \, d\tau \]

charge flowing outward through surface decrease in the amount of charge remaining in \( V \)

So that for any volume,

\[ \nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} \]

Equation of continuity

This is just an expression of the conservation of charge.

### 2.2 Biot-Savart law

**Who were they?**

Biot: French physicist (1774-1862) who worked on optics and took a lot of scientific apparatus up in a hot air balloon with fellow physicist Gay-Lussac...

Savart: French professor (1791-1841) at the College de France. Collaborated with Biot.

**What does the law do for us?**

Tells us how to calculate the magnetic induction \( \mathbf{B} \) due to steady currents.

We now use the **Biot-Savart law** to deal with problems in magnetostatics: this is the situation of steady currents leading to constant magnetic fields.
We must consider *extended current distributions* (cf. electrostatics where you considered point charges; a moving point charge *cannot* produce a constant magnetic field!)

Consider a current carrying wire in an **arbitrary** geometry:

The Biot-Savart law gives us:

\[
B = \frac{\mu_0}{4\pi} \int_0^\infty \frac{dl \times r}{r^2}
\]

The integral is carried out over the closed current path.

*dl* is an integration element of length along the current path.
**r** is a position vector pointing from the element of circuit **dl** towards the field point at which we want to know **B**

The magnetic induction **B** is measured in SI units teslas (T)

The constant \( \mu_0 = 4\pi \times 10^{-7} \) tesla metre/ampere is called the **permeability of free space**.

[Don’t be afraid of \( \mu_0 \) – ‘permeability of free space’ is rather olde world English! All \( \mu_0 \) really does is fix the ‘strength’ of the magnetic field: it *determines the intensity of the magnetic induction*. Just look at the formula.]

### 2.2.1 How big is 1 tesla?

**B** is often given in the non-SI unit of gauss: \( 1 \text{T} = 10^4 \) gauss.

Earth’s magnetic field is \( \sim 1/2 \) gauss so \( 1 \text{T} \) is \( \sim 10,000 \) x Earth’s field.
For comparison,

A small bar magnet will produce $B \sim 10^{-2}$ T
MRI body scanner magnet $B \sim 2$ T
A hair dryer $B \sim 10^{-7}$ –$10^{-3}$ T
Colour TV $\sim 10^{-6}$ T
Magnets in the School of Physics research labs produce up to $\sim 50$T
At a sunspot $B \sim 0.3$ T
Sunlight $B \sim 3 \times 10^{-6}$ T
The field at the surface of a neutron star is thought to be $\sim 10^8$ T.
2.2.2 Biot Savart Law: Applications

1. B due to a circular current carrying loop

\[
\begin{align*}
 dB &= \mu_0 I \frac{dl \hat{r}}{r^2} \\
 B &= \mu_0 I \int \frac{dl \hat{r}}{r^2}
\end{align*}
\]
The total magnetic induction is along the axis (symmetry – the components pointing in directions perpendicular to the loop’s axis sum to zero) and we see that

\[ dB_z = \frac{\mu_0 I 2\pi a}{4\pi r^2} \cos \theta \]

\[ = \frac{\mu_0 I a^2}{2(a^2 + z^2)^{3/2}} \]

B on axis of a circular current loop, radius a

[see Griffiths p 218 and Tipler p887 (full details) ]

2. B around a long straight current carrying wire
The Biot-Savart law is

\[ B = \frac{\mu_0 I}{4\pi} \oint \frac{dl \times r}{r^2} \]

Now \( dl \times r = dl \sin \theta \) and \( dB \) points in the azimuthal direction (\( \hat{\phi} \)) around the wire. The right hand rule gives the direction of \( \hat{\phi} \).

So that

\[ dB = \frac{\mu_0 I dl \sin \theta}{4\pi} \frac{r}{r^2} \hat{\phi} \]

Now express \( dl \), \( \sin \theta \) and \( r \) in terms of \( \alpha \) and \( s \), and (see Tipler 892)

\[ B = \frac{\mu_0 I}{4\pi} \int_{-\pi/2}^{\pi/2} \cos \alpha d\alpha \hat{\phi} \]

\[ = \frac{\mu_0 I}{2\pi s} \hat{\phi} \]

\( s \) is radial distance from wire
This is consistent with what we already know about the field around a straight wire:

3. Force of attraction between parallel current carrying wires

Consider two infinite parallel wires spaced d metres apart:

wire 1. exerts a force (directed to the left) on wire 2.
Using the result from 2. above, the magnetic induction at radial distance \( s \) from an infinite wire is

\[
B = \frac{\mu_0 I}{2\pi s}
\]

So wire 1. produces a magnetic induction \( B \) at wire 2. of

\[
B = \frac{\mu_0 I}{2\pi d}
\]

spacing between the wires

Using the result from section 2.1.2 above,

\[
F_m = \int I (dl \times B)
\]

Force on a current-carrying wire in field \( B \)

we have

\[
F = I_2 \left( \frac{\mu_0 I_1}{2\pi d} \right) \int dl
\]

B field due to current in wire I

Since we have chosen infinitely long wires, the \( \int dl \) gives an infinite force!
However, the force per unit length is finite:

\[ F \text{ per unit length} = \frac{\mu_0 I_1 I_2}{2\pi d} \]

For parallel currents: force is attractive

For anti-parallel currents: force is repulsive

4. Moving charges: magnetic and electric forces

Here is an interesting problem using the result we have just derived!

Consider two moving infinite lines of charge, \( \lambda \) coulombs per metre:
The magnetic induction due to a straight wire is

\[ B = \frac{\mu_0 I}{2\pi r} \]

so the field at wire 2 due to wire 1 is

\[ B_2 = \frac{\mu_0 I_1}{2\pi r} \]

resulting in a force (\(dF = Idl \times B\))

\[ F = I_2 \left( \frac{\mu_0 I_1}{2\pi a} \right) \int dl \]

giving a magnetic force *per unit length*

\[ F_m = \frac{\mu_0 I_1 I_2}{2\pi a} \]

and with \(I_1 = I_2 = \lambda v\),

\[ F_m = \frac{\mu_0 \lambda^2 v^2}{2\pi a} \]

The electric field of one line of charge is

\[ E = \frac{1}{2\pi \varepsilon_0} \frac{\lambda}{s} \]

giving an electric repulsion per unit length of one wire on the other

\[ F_e = \frac{1}{2\pi \varepsilon_0} \frac{\lambda^2}{a} \]
[E=F/q]. Equating $F_m$ and $F_E$,

$$F_m = \frac{\mu_0 \lambda^2 v^2}{2\pi a} = F_e = \frac{1}{2\pi\varepsilon_0} \frac{\lambda^2}{a}$$

$$v^2 = \frac{1}{\varepsilon_0 \mu_0}$$

$$v = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = \frac{1}{\sqrt{(8.85 \times 10^{-12}) (4\pi \times 10^{-7})}}$$

$$= 2.998 \times 10^8 \text{ ms}^{-1}$$

Therefore, the electric and magnetic forces are equal when $v = c$! according to this calculation.

**Remember!** The drift velocity of electrons in a conductor is typically in the range mm.$s^{-1}$ to cm.$s^{-1}$ only – it is surprisingly small.