# 2. CURRENTS AND THE BIOT-SAVART LAW

## 2.1 Electric currents

An electric current is a movement of charge along a line (a wire), across a surface (a conducting sheet) or in a volume. We recall that

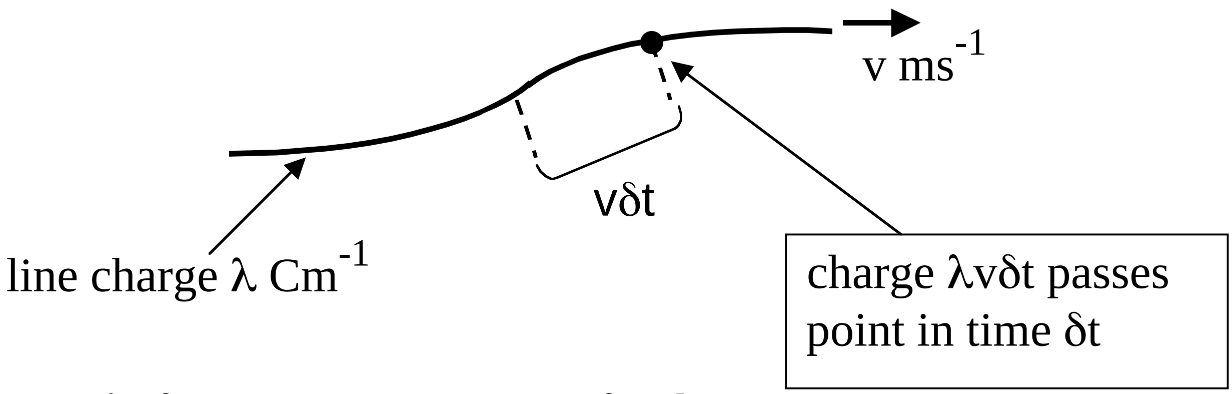
$$I = dq/dt \quad \text{and} \quad I = nAve$$

current is in amperes (A) or coulombs per second ($Cs^{-1}$)

Current in a conducting wire of cross-sectional area A with n free electrons per $m^3$ and drift velocity v $ms^{-1}$

### 2.1.1 Line of charge

A current in a wire can be considered a line of charge of linear charge density $\lambda$ moving at v $ms^{-1}$.



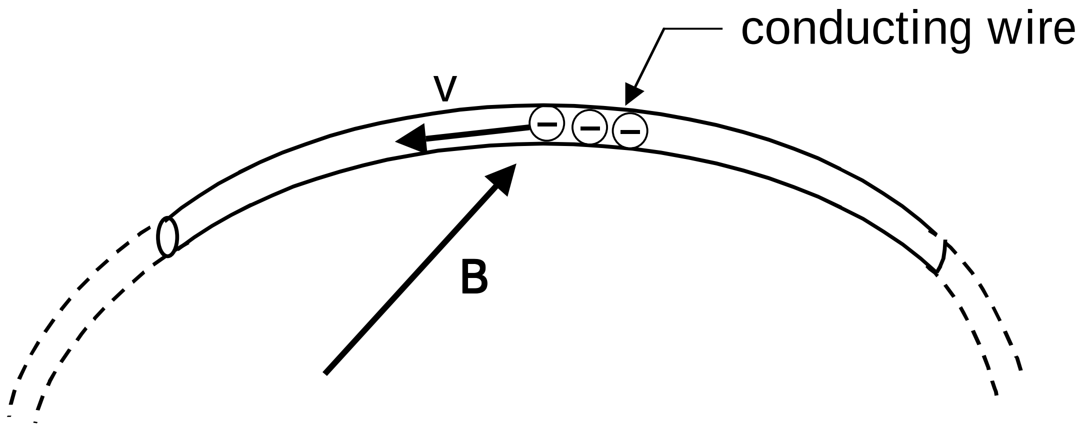

If charge $\lambda v\delta t$ passes point in time $\delta t$, then

$$\mathbf{I} = \lambda\mathbf{v}$$

velocity

Note: we have identified current as *a vector* (see p209 of Griffiths).

### 2.1.2 Force on current-carrying wire




For a steady current and fixed magnetic field,

$$F_m = \int(\mathbf{v}\mathbf{x}\mathbf{B})dq$$

and

$$dq = \lambda dl$$

so that

$$F_m = \int(\mathbf{v}\mathbf{x}\mathbf{B})\lambda dl$$

$$= \int(\mathbf{I}\mathbf{x}\mathbf{B})dl \qquad (\mathbf{I} \text{ and } \mathbf{v} \text{ are in same dir'n})$$

$$= \int I(d\mathbf{l}\mathbf{x}\mathbf{B}) \qquad (\mathbf{I} \text{ and } d\mathbf{l} \text{ are in same dir'n})$$

For current that is constant in magnitude,

$$F_m = I\int(d\mathbf{l}\mathbf{x}\mathbf{B})$$

### 2.1.3 Surface and volume currents

When a surface charge density σ moves with velocity v over a surface we have a **surface current** and **surface current density K**:

$$\mathbf{K} = \sigma\mathbf{v}$$

and

$$F_m = \int(\mathbf{v}\mathbf{x}\mathbf{B})\sigma da = \int(\mathbf{K}\mathbf{x}\mathbf{B})da$$

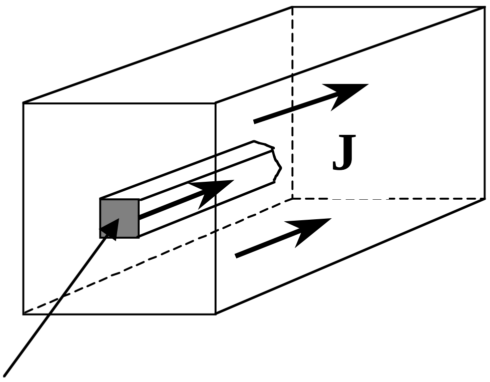
Similarly, when a current flows in some volume, we consider the **volume charge density** ρ and **volume current density J** and

$$\mathbf{J} = \rho\mathbf{v}$$

and the magnetic force is

$$F_m = \int(\mathbf{v}\mathbf{x}\mathbf{B})\rho d\tau = \int(\mathbf{J}\mathbf{x}\mathbf{B})d\tau$$

dτ is a volume element




$$\mathbf{J} = \frac{d\mathbf{I}}{da_\perp}$$

**J** is the current per unit area perpendicular to the flow

$da_\perp$ is an infinitesimal cross-section perpendicular to the current flow

## 2.1.4 Equation of continuity

Using

$$\mathbf{J} = \frac{d\mathbf{I}}{da_\perp}$$

we can write

$$I = \int_S J da_\perp = \int_S \mathbf{J.da}$$

Using the divergence theorem (see the inside cover of Griffiths) we see that the total charge leaving volume V per unit time is

$$\oint_S \mathbf{J}.d\mathbf{a} = \int_V (\nabla.\mathbf{J}) d\tau$$

surface integral over closed surface S

(see Griffith, p32 for a discussion of the geometrical interpretation of the divergence theorem)

Now

$$\underbrace{\int_V (\nabla.\mathbf{J})d\tau}_{\text{charge flowing outward through surface}} = \underbrace{-\frac{d}{dt}\int_V \rho d\tau}_{\text{decrease in the amount of charge remaining in V}} = -\int_V \frac{\partial \rho}{\partial t}d\tau$$

So that for any volume,

$$\nabla.\mathbf{J} = -\frac{\partial \rho}{\partial t} \qquad \text{Equation of continuity}$$

This is just an expression of the conservation of charge.

## 2.2 Biot-Savart law

**Who were they?**

Biot: French physicist (1774-1862) who worked on optics and took a lot of scientific apparatus up in a hot air balloon with fellow physicist Gay-Lussac...

Savart: French professor (1791-1841) at the College de France. Collaborated with Biot.

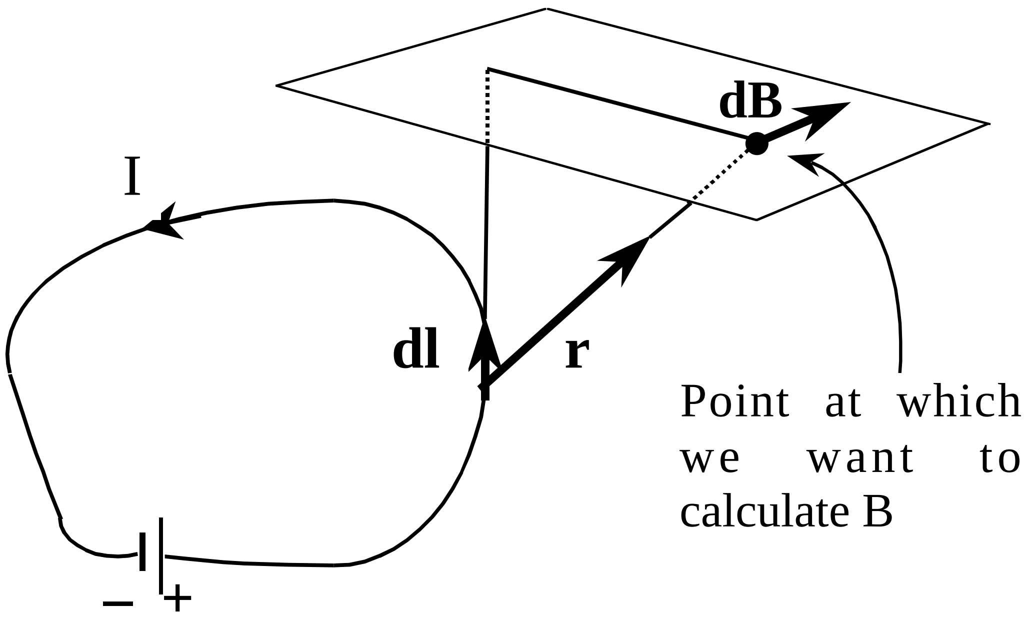
**What does the law do for us?**

Tells us how to calculate the magnetic induction **B** due to steady currents.

We now use the **Biot-Savart law** to deal with problems in **magnetostatics**: this is the situation of **steady** currents leading to **constant** magnetic fields.

We must consider *extended current distributions* (cf. electrostatics where you considered point charges; a moving point charge *cannot* produce a constant magnetic field!)

Consider a current carrying wire in an **arbitrary** geometry:




The Biot-Savart law gives us:

$$\mathbf{B} = \frac{\mu_0}{4\pi} I \oint \frac{\mathbf{dl x r}}{r^2}$$

The integral is carried out over the closed current path

**dl** is an integration element of length along the current path

**r** is a position vector pointing from the element of circuit **dl** *towards* the field point at which we want to know **B**

The magnetic induction **B** is measured in SI units teslas (T)

The constant $\mu_0 = 4\pi x 10^{-7}$ tesla metre / ampere is called the **permeability of free space**.

[Don't be afraid of $\mu_0$ – 'permeability of free space' is rather olde world English! All $\mu_0$ really does is fix the 'strength' of the magnetic field: it *determines the intensity of the magnetic induction.* Just look at the formula.]

### 2.2.1 How big is 1 tesla?

**B** is often given in the non-SI unit of gauss: 1T = $10^4$ gauss.

Earth's magnetic field is ~ 1/2 gauss so 1T is ~ 10,000 x Earth's field.

**For comparison,**

A small bar magnet will produce B ~$10^{-2}$ T

MRI body scanner magnet B ~ 2T

A hair dryer B ~ $10^{-7}$ –$10^{-3}$ T

Colour TV ~ $10^{-6}$ T

Magnets in the School of Physics research labs produce up to ~50T

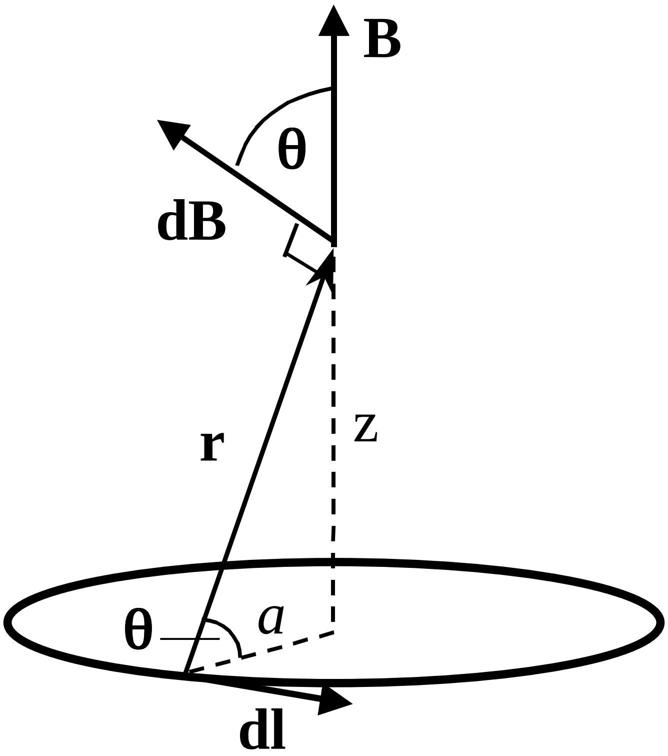
At a sunspot B ~ 0.3 T

Sunlight B ~ 3 x $10^{-6}$ T

The field at the surface of a neutron star is thought to be ~$10^{8}$ T.

## 2.2.2 Biot Savart Law: Applications

### 1. B due to a circular current carrying loop




$$\mathbf{dB} = \frac{\mu_0 I}{4\pi} \frac{\mathbf{dl x \hat{r}}}{r^2}$$

and

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \oint \frac{\mathbf{dl x \hat{r}}}{r^2}$$

The total magnetic induction is along the axis (symmetry – the components pointing in directions perpendicular to the loop's axis sum to zero) and we see that

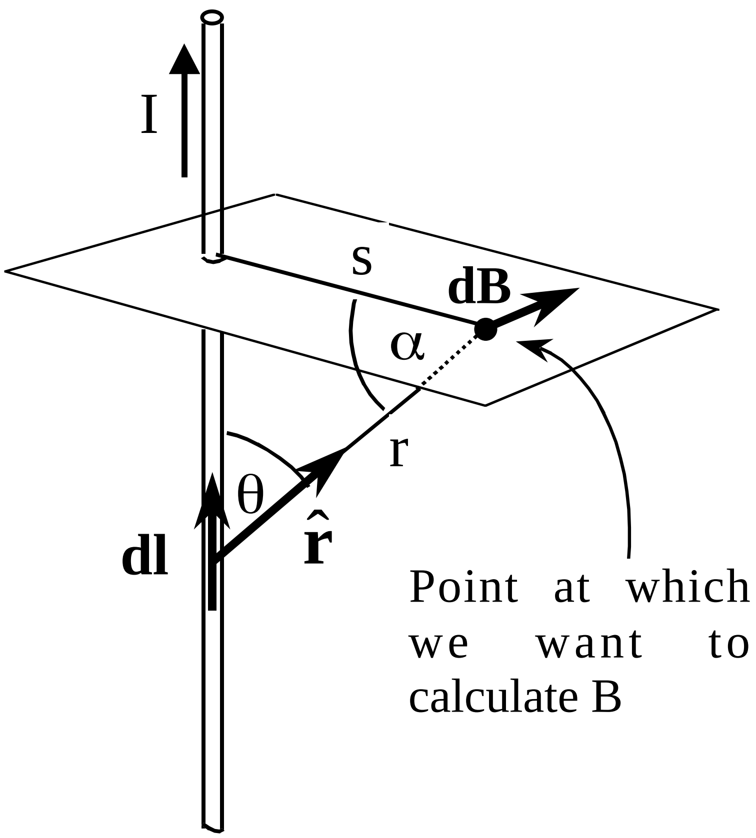
$$dB_z = \frac{\mu_0 I}{4\pi}\frac{2\pi a}{r^2}\cos\theta$$

$$= \frac{\mu_0 I a^2}{2(a^2+z^2)^{3/2}}$$

B on axis of a circular current loop, radius a

[see Griffiths p 218 and Tipler p887 (full details) ]

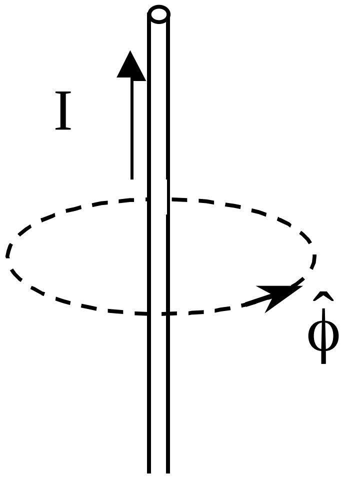
**2. B around a long straight current carrying wire**



The Biot-Savart law is

$$\mathbf{B} = \frac{\mu_0}{4\pi} I \oint \frac{\mathbf{dlxr}}{r^2}$$

Now $\mathbf{dlxr} = dl \sin\theta$ and dB points in the azimuthal direction ($\hat{\phi}$) around the wire. The right hand rule gives the direction of $\hat{\phi}$.




So that

$$\mathbf{dB} = \frac{\mu_0 I}{4\pi} \frac{dl \sin\theta}{r^2} \quad \hat{\phi}$$

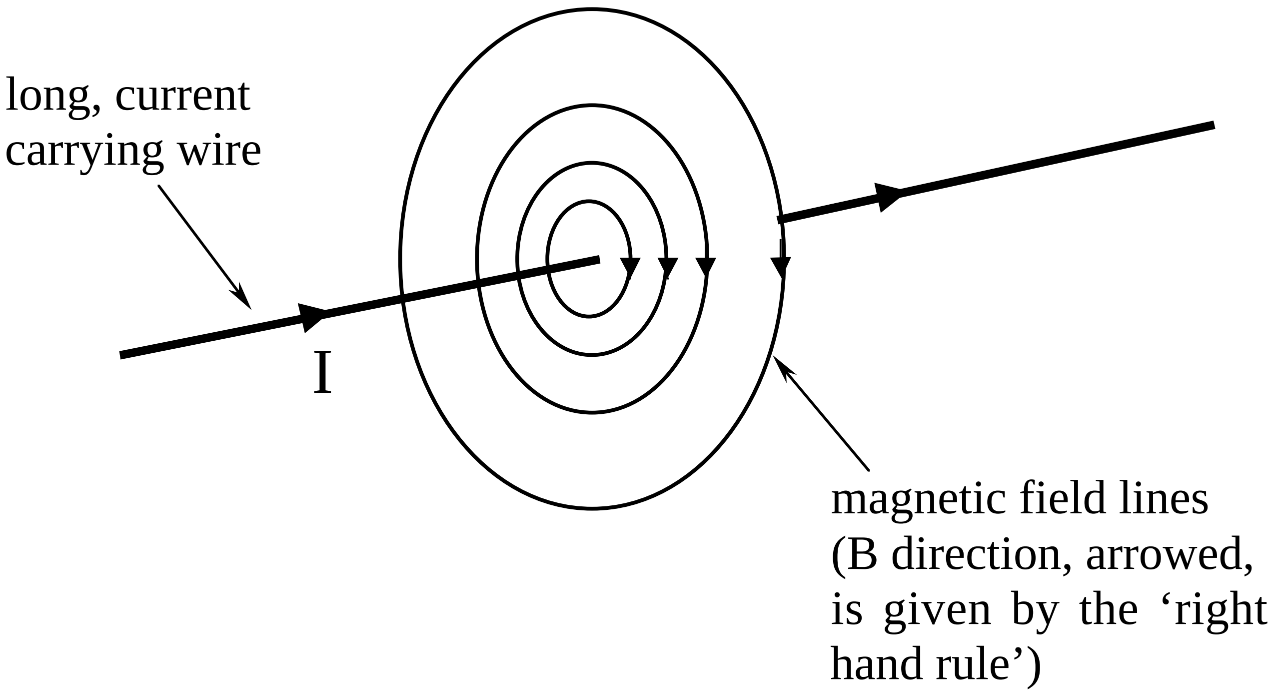
Now express dl, sin$\theta$ and r in terms of $\alpha$ and s, and (see Tipler 892)

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \int_{-\pi/2}^{\pi/2} \cos\alpha \, d\alpha \ \hat{\phi}$$

$$= \frac{\mu_0 I}{2\pi s} \hat{\phi}$$

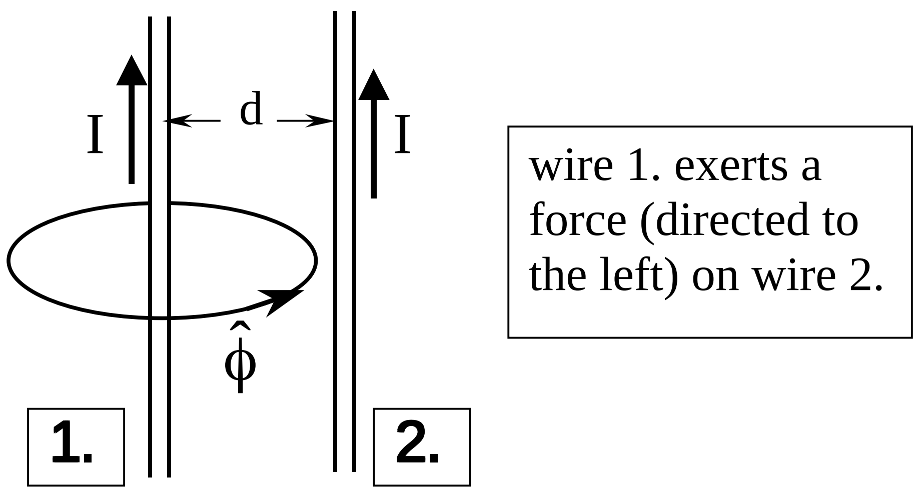
s is radial distance from wire

This is consistent with what we already know about the field around a straight wire:




**3. Force of attraction between parallel current carrying wires**

Consider two *infinite* parallel wires spaced d metres apart:



Using the result from 2. above, the magnetic induction at radial distance s from an infinite wire is

$$B = \frac{\mu_0 I}{2\pi s}$$

So wire 1. produces a magnetic induction B at wire 2. of

$$B = \frac{\mu_0 I}{2\pi d}$$

spacing between the wires

Using the result from section 2.1.2 above,

$$\mathbf{F_m} = \int I(d\mathbf{l} \mathbf{x} \mathbf{B})$$

Force on a current-carrying wire in field B

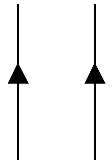
we have

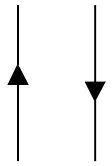
$$F = I_2 \underbrace{\left(\frac{\mu_0 I_1}{2\pi d}\right)}_{\text{B field due to current in wire I}} \int dl$$

Since we have chosen infinitely long wires, the $\int dl$ gives an infinite force!

However, the force per unit length *is* finite:

$$\text{F per unit length} = \frac{\mu_0}{2\pi}\frac{I_1 I_2}{d}$$

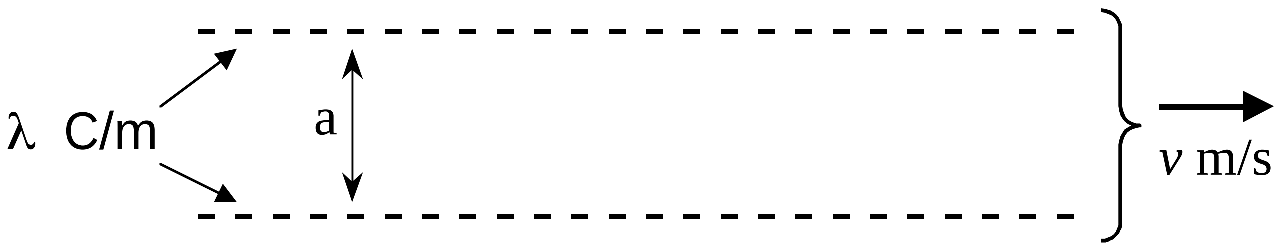
For *parallel* currents:  force is *attractive*

For *anti-parallel* currents:  force is *repulsive*

**4. Moving charges: magnetic and electric forces**

Here is an interesting problem using the result we have just derived!

Consider two moving infinite lines of charge, λ coulombs per metre:



The magnetic induction due to a straight wire is

$$B = \frac{\mu_0 I}{2\pi r}$$

so the field at wire 2 due to wire 1 is

$$B_2 = \frac{\mu_0 I_1}{2\pi r}$$

resulting in a force (dF = IdlxB)

$$F = I_2\left(\frac{\mu_0 I_1}{2\pi a}\right)\int dl$$

giving a magnetic force *per unit length*

$$F_m = \frac{\mu_0}{2\pi}\frac{I_1 I_2}{a}$$

and with $I_1 = I_2 = \lambda v$,

$$F_m = \frac{\mu_0}{2\pi}\frac{\lambda^2 v^2}{a}$$

The electric field of one line of charge is

$$E = \frac{1}{2\pi\varepsilon_0}\frac{\lambda}{s}$$

giving an electric repulsion per unit length of one wire on the other

$$F_e = \frac{1}{2\pi\varepsilon_0}\frac{\lambda^2}{a}$$

[ E=F/q]. Equating $F_m$ and $F_E$,

$$F_m = \frac{\mu_0}{2\pi}\frac{\lambda^2 v^2}{a} = F_e = \frac{1}{2\pi\varepsilon_0}\frac{\lambda^2}{a}$$

$$v^2 = \frac{1}{\varepsilon_0 \mu_0}$$

$$v = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = \frac{1}{\sqrt{(8.85x10^{-12})(4\pi x10^{-7})}}$$

$$= 2.998x10^8 \ ms^{-1}$$

Therefore, the electric and magnetic forces are equal when v = c! according to this calculation.

**Remember!** The drift velocity of electrons in a conductor is typically in the range $mm.s^{-1}$ to $cm.s^{-1}$ only – it is surprisingly small.