THE THEORY OF SPACE, TIME AND GRAVITATION

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Let us now consider the geometrical meaning of the equations $x_0 =$ const. and $x_k =$ const. We shall derive a condition under which the equation

$$\omega(x, y, z, t) = 0 \tag{35.30}$$

can be interpreted as the equation of a surface in motion. It follows from this equation that the differentials of space and time coordinates, are related by

$$\omega_x\, dx + \omega_y\, dy + \omega_z\, dz + \omega_t\, dt = 0 \tag{35.31}$$

where ω_x, ω_y, ω_z and ω_t denote the derivatives of ω with respect to x, y, z and t. We take a displacement (dx, dy, dz) in the direction of the normal to the surface and put

$$dx = \frac{\omega_x}{|\text{grad}\,\omega|}\, dn\,; \qquad dy = \frac{\omega_y}{|\text{grad}\,\omega|}\, dn\,; \qquad dz = \frac{\omega_z}{|\text{grad}\,\omega|}\, dn \tag{35.32}$$

so that $|dn|$ is the absolute value of the displacement. Inserting into (35.31) we get

$$|\text{grad}\,\omega|\, dn + \omega_t dt = 0 \tag{35.33}$$

and therefore the square of the displacement velocity

$$v^2 = \left(\frac{dn}{dt}\right)^2 \tag{35.34}$$

will be given by

$$v^2 = \frac{\omega_t^2}{(\text{grad}\,\omega)^2} \tag{35.35}$$

Thus (35.30) can be interpreted as the equation of a surface, each point of which moves normally with a speed given by (35.35). However, such an interpretation is only possible as long as this speed does not exceed that of light. According to (35.35) and (35.01) this means that we must have

$$(\nabla\omega)^2 \leqslant 0 \tag{35.36}$$

The equality sign is valid for motion with the speed of light.

On the other hand, if

$$(\nabla\omega)^2 > 0 \tag{35.37}$$

equation (35.30) can be solved for the time and written in the form

$$t = \frac{1}{c} f(x, y, z) \tag{35.38}$$

with

$$(\text{grad}\, f)^2 < 1 \tag{35.39}$$

Equation (35.38) assigns to every point in space a definite instant of time in such a way that all the four-dimensional "point-instants" are quasi-simultaneous. Such an equation may be called a "time-equation". We recall that time equations occurred in Section 3 in connection with the question of the characteristics of Maxwell's equations.

As we remarked in Section 3, an equation $\omega = 0$ can be considered as the equation of a hypersurface in the four-dimensional space-time manifold. Such hypersurfaces can then be divided into two classes.

If $(\nabla\omega)^2 < 0$ we can say that one of the dimensions of the hypersurface is time-like (the inaccurate phrase "the surface is time-like" is sometimes used).

ω IS IMAGINARY, and therefore "time-like"

ω IS REAL, and therefore "space-like"

By (35.35) this describes an ordinary two-dimensional surface† moving with a velocity less than that of light.

If $(\nabla\omega)^2 > 0$, on the other hand, we say that the hypersurface is space-like. It then represents the whole of infinite space, the various points of which are all taken at different instants of time, the time t at which the point (x, y, z) is taken, being determined by the time equation, i.e. the equation of the hypersurface; the instants of time assigned to any two points in space must be so close that the corresponding four-dimensional interval is always space-like.

We use the fact that $(\nabla\omega)^2$ is an invariant and in turn put $\omega = x_0$, $\omega = x_1$, $\omega = x_2$ and $\omega = x_3$. This gives

$$(\nabla x_0)^2 = g^{00} > 0 \tag{35.40}$$

and $$(\nabla x_1)^2 = g^{11} < 0\,; \quad (\nabla x_2)^2 = g^{22} < 0\,; \quad (\nabla x_3)^2 = g^{33} < 0 \tag{35.41}$$

Hence the equation $x_0 =$ const. is a time equation and the three equations $x_k =$ const. $(k = 1, 2, 3)$ represent surfaces moving in the direction of their normals with less than light velocity. These latter are thus equations of moving spatial coordinate surfaces.

It follows also from our conditions on the transformations of space and time coordinates that constant values of x_1, x_2 and x_3 correspond, in any inertial frame of reference, to motion of a point with less than light velocity.

In classical Newtonian mechanics one often uses a time dependent coordinate transformation which is interpreted as passing to a moving frame of reference. In comparing coordinate transformations in Newtonian mechanics with the transformations of space and time coordinates in the Theory of Relativity it is essential to realize the following. Firstly, in the general case of accelerated motion the very notion of a frame of reference in Newtonian mechanics is not the same as in Relativity. The Newtonian concept involves the idea of an absolutely rigid body and the instantaneous propagation of light. In Relativity, on the other hand, the notion of a rigid body is used, if at all, not in an absolute sense but only for non-accelerated motions and in the absence of external forces, and is of an auxiliary nature; the concept of a frame of reference is not based on it but on the law of wave front propagation. The prototype of a Newtonian frame of reference is a rigid scaffolding, the prototype of a Relativistic one is the radar station. Secondly, the class of transformations permissible in Newtonian mechanics is much wider than in the Theory of Relativity; Newtonian mechanics does not have to consider the limitations, discussed above, which arise from the existence of a limiting speed.

As an example we consider a transformation which can be interpreted in Newtonian mechanics as going over to a uniformly accelerated frame. Let x', y', z' and t' be the space and time coordinates in an inertial frame, i.e. Galilean coordinates. We put

$$x' = x - \frac{1}{2}at^2\,; \quad y' = y\,; \quad z' = z \tag{35.42}$$

and also $$t' = t - \frac{a}{c^2}tx. \tag{35.43}$$

† In the four-dimensional manifold a hypersurface has three dimensions but in the present case only two of these are spatial.

The variables x, y, z and t can be interpreted as space and time coordinates in a certain accelerated frame (in the Newtonian sense and in the corresponding approximation). Inserting (35.42) and (35.43) into the expression for ds^2 we get

$$ds^2 = (c^2 - 2ax - a^2t^2)\,dt^2 - dx^2 - dy^2 - dz^2 + \frac{a^2}{c^2}(x\,dt + t\,dx)^2 \qquad (35.44)$$

The required inequalities for the coefficients will hold if the conditions

$$1 - \frac{a^2t^2}{c^2} > 0\,; \qquad \left(1 - \frac{ax}{c^2}\right)^2 - \frac{a^2t^2}{c^2} > 0 \qquad (35.45)$$

are satisfied. In addition we can require that

$$\frac{\partial t'}{\partial t} = 1 - \frac{ax}{c^2} > 0 \qquad (35.46)$$

These inequalities show that the substitutions (35.42), (35.43) are permissible only in a part of space and only for a limited length of time.

Another example is the transformation corresponding to the introduction of a uniformly rotating frame. We put

$$\begin{aligned} x' &= x\cos\omega t + y\sin\omega t\,; & z' &= z \\ y' &= -x\sin\omega t + y\cos\omega t\,; & t' &= t \end{aligned} \qquad (35.47)$$

and obtain

$$ds^2 = [c^2 - \omega^2(x^2 + y^2)]\,dt^2 - 2\omega(y\,dx - x\,dy)\,dt - dx^2 - dy^2 - dz^2 \qquad (35.48)$$

The conditions on the coefficients require

$$c^2 - \omega^2(x^2 + y^2) > 0 \qquad (35.49)$$

which is satisfied only for distances from the axis of rotation less than that where the linear velocity of the rotation equals the speed of light.

We stress once again that the examples given here have physical sense only in a region in which Newtonian mechanics is applicable (see also Section 61).

It is obvious that the introduction of ordinary curvilinear spatial coordinates is always an allowed transformation. As long as the transformations do not involve time they have the same geometrical meaning as in non-relativistic theory. Therefore we refrain from discussing them.

36. General Tensor Analysis and Generalized Geometry

In the previous section we considered the expressions

$$(\nabla\omega)^2 = \sum_{\alpha,\beta=0}^{3} g^{\alpha\beta}\frac{\partial\omega}{\partial x_\alpha}\frac{\partial\omega}{\partial x_\beta} \qquad (36.01)$$

and

$$ds^2 = \sum_{\alpha,\beta=0} g_{\alpha\beta}\,dx_\alpha\,dx_\beta \qquad (36.02)$$

which were obtained from the usual expressions of Relativity Theory by introducing variables x_1, x_2, x_3 and x_0 in place of the space and time coordinates x, y, z and t. We established the conditions subject to which the variable x_0 can characterize a sequence of events in time and the variables x_1, x_2 and x_3 their location in space.

REMOVES IMAGINARY ANGULAR VELOCITY ω FROM CONSIDERATION