

8. MAXWELL'S EQUATIONS

So far we have seen the four equations:

$$(i) \quad \nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \rho \quad (\text{Gauss's law})$$

$$(ii) \quad \nabla \cdot \mathbf{B} = 0 \quad (\text{no name, just } \text{div} \mathbf{B} \text{ equals zero!})$$

$$(iii) \quad \nabla \times \mathbf{E} = - \frac{\partial \mathbf{B}}{\partial t} \quad (\text{Faraday's law})$$

$$(iv) \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \quad (\text{Ampère's law})$$

that together are *almost* Maxwell's equations aside...we need to include the displacement current.

8.2 Displacement current

Up to now we have written

$$\nabla \times \mathbf{H} = \mathbf{J}$$

which would be one of the 'Maxwell equations' but there is a term missing (in fact it was Maxwell himself who derived this term to 'fix-up' the set of four Maxwell equations giving the correct description of EM).

If we take the divergence of both sides of $\nabla \times \mathbf{H} = \mathbf{J}$ we find

$$\nabla \cdot (\nabla \times \mathbf{H}) = 0 = \nabla \cdot \mathbf{J} \quad (*)$$



div of curl of any vector field is zero
(see vector identity (9) inside front
cover of Griffiths)

But the Continuity eqn says

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$$

for time-varying fields so (*) cannot be correct in general. To correct things we need to add the time-varying term $\frac{\partial \rho}{\partial t}$ to the r.h.s of (*):

$$\nabla \cdot (\nabla \times \mathbf{H}) = 0 = \nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t}$$

and since $\nabla \cdot \mathbf{D} = \rho$,

$$\nabla \cdot (\nabla \times \mathbf{H}) = 0 = \nabla \cdot \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right)$$

or,

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

‘corrected’ Maxwell
equation for $\text{curl} \mathbf{H}$

The time-varying term $\frac{\partial \mathbf{D}}{\partial t}$ is called the **displacement current**.

The equation

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

tells us that there will be a magnetic field (due to the displacement current $\partial \mathbf{D} / \partial t$) *even when there is no current flow* (i.e. when $\mathbf{J} = 0$).

We usually write $\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$ in terms of \mathbf{B} and \mathbf{E} :

$$\begin{array}{ccc} \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} & & \\ \swarrow & & \searrow \\ \text{by recalling } \mathbf{B} = \mu_0 \mathbf{H} \text{ (free space) and } \mathbf{D} = \epsilon_0 \mathbf{E} \text{ (free space)} & & \\ \swarrow & & \searrow \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} & & \end{array}$$

8.2 Maxwell's Equations in differential form

We can now write the complete set of Maxwell equations, including the correction term (the displacement current):

$$(i) \quad \nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \rho \quad (\text{Gauss's law})$$

$$(ii) \quad \nabla \cdot \mathbf{B} = 0 \quad (\text{no name, just div}\mathbf{B} \text{ equals zero!})$$

$$(iii) \quad \nabla \times \mathbf{E} = - \frac{\partial \mathbf{B}}{\partial t} \quad (\text{Faraday's law})$$

$$(iv) \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad (\text{Ampère's law with Maxwell's correction})$$

Eqns (i) – (iv) tell us how CHARGES produce FIELDS

and the force equation (electric + magnetic force acting on a moving charge)

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

tells us how FIELDS affect CHARGES which together with the equation of continuity

$$\nabla \cdot \mathbf{J} = - \frac{\partial \rho}{\partial t}$$

provide all the mathematical apparatus needed to describe electromagnetism – that is to solve all problems in classical electromagnetism on the macroscopic scale.

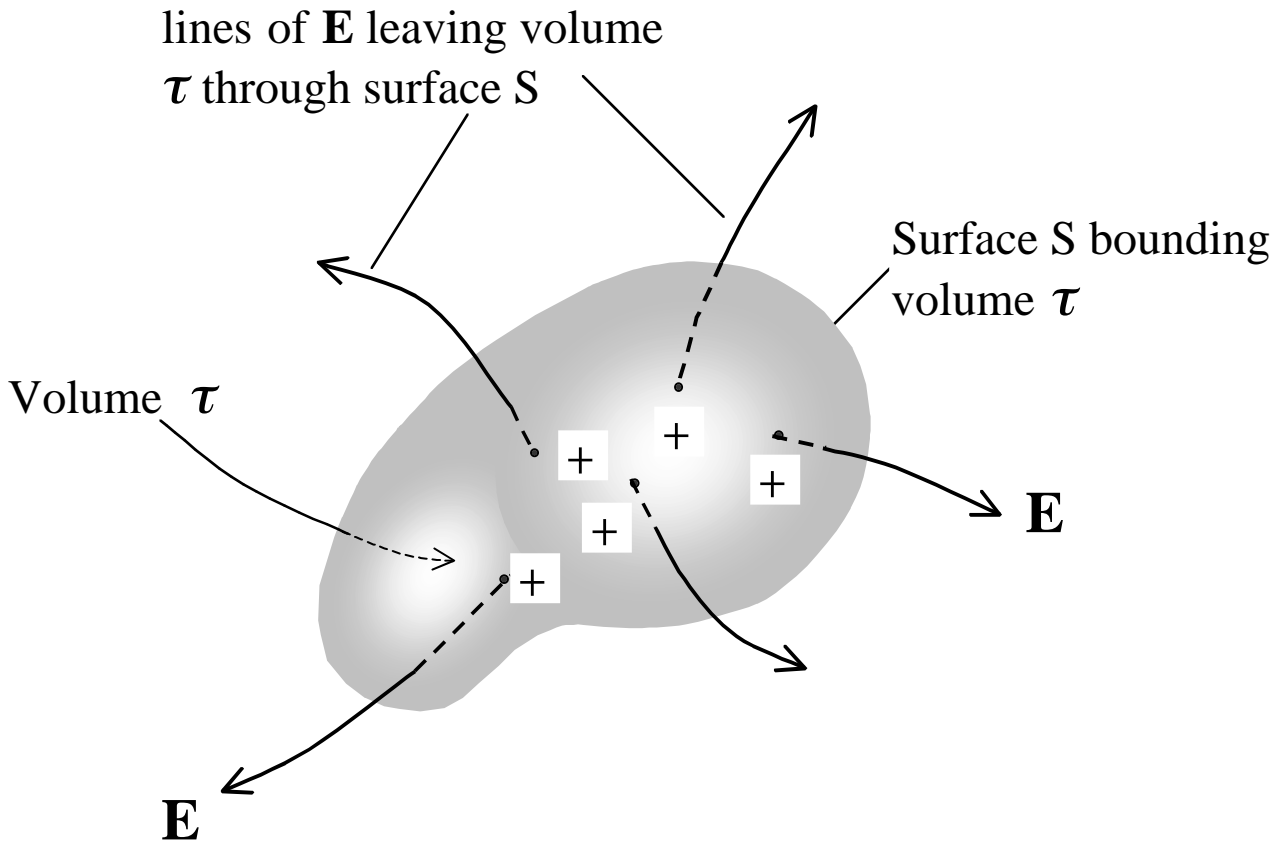
8.3 Maxwell's equations in integral form

Maxwell's equations in integral form perhaps give us a greater physical insight:

$$\begin{array}{c}
 \text{(i) } \oint_S \mathbf{D} \cdot d\mathbf{a} = Q_{f,enc} \\
 \uparrow \\
 \text{integrate over closed surface } S \\
 \downarrow \\
 \text{(ii) } \oint_S \mathbf{B} \cdot d\mathbf{a} = 0 \\
 \\
 \text{(iii) } \oint_L \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \oint_S \mathbf{B} \cdot d\mathbf{a} \\
 \begin{array}{cc}
 \uparrow & \uparrow \\
 \text{closed loop } L & \text{bounding surface } S
 \end{array} \\
 \begin{array}{cc}
 \downarrow & \downarrow \\
 \text{(iv) } \oint_L \mathbf{H} \cdot d\mathbf{l} = I_{f,enc} + \frac{d}{dt} \oint_S \mathbf{D} \cdot d\mathbf{a}
 \end{array}
 \end{array}$$

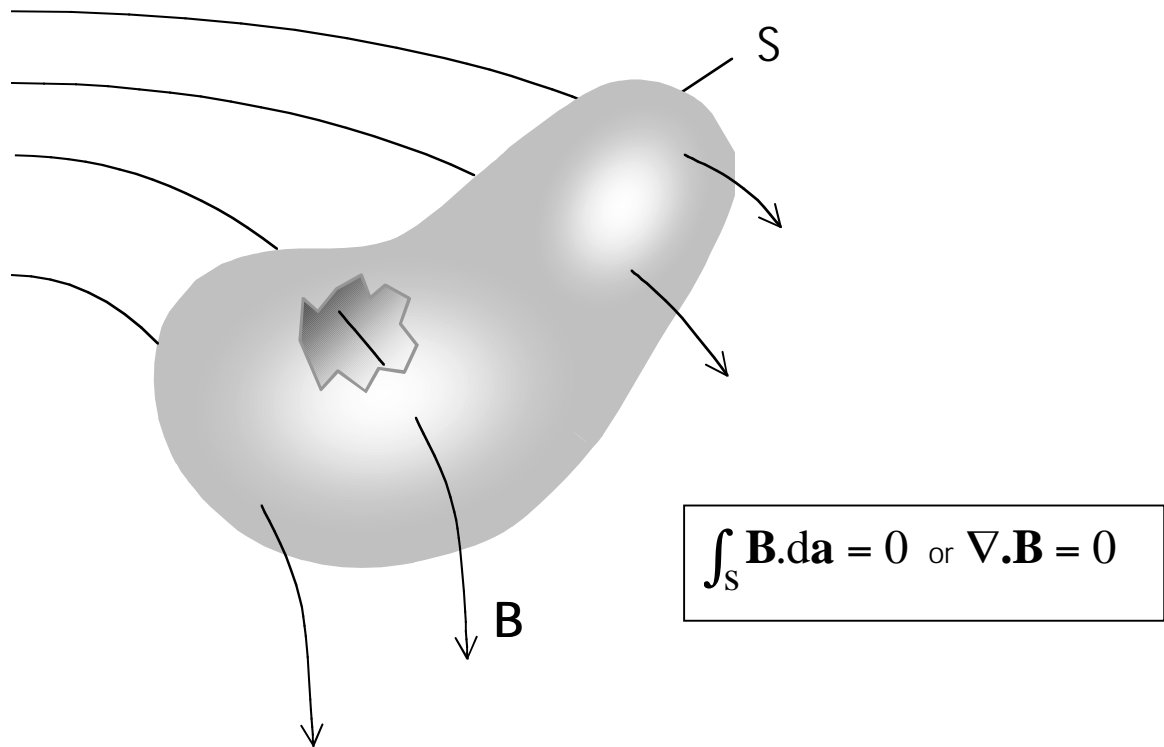
8.4 Visualization of Maxwell's equations

(i) Gauss' law: $\oint_S \mathbf{D} \cdot d\mathbf{a} = Q_{f,enc}$ or $\oint_S \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{total}}{\epsilon_0}$



Lines of \mathbf{E} begin on positive charges. \mathbf{E} lines exit enclosing volume τ through surface S . Gauss' law says the total flux of \mathbf{E} leaving enclosed volume τ is equal to the total charge enclosed by surface S divided by ϵ_0 .

(ii) $\nabla \cdot \mathbf{B} = 0$ or $\oint_S \mathbf{B} \cdot d\mathbf{a} = 0$
 (differential form) (integral form)



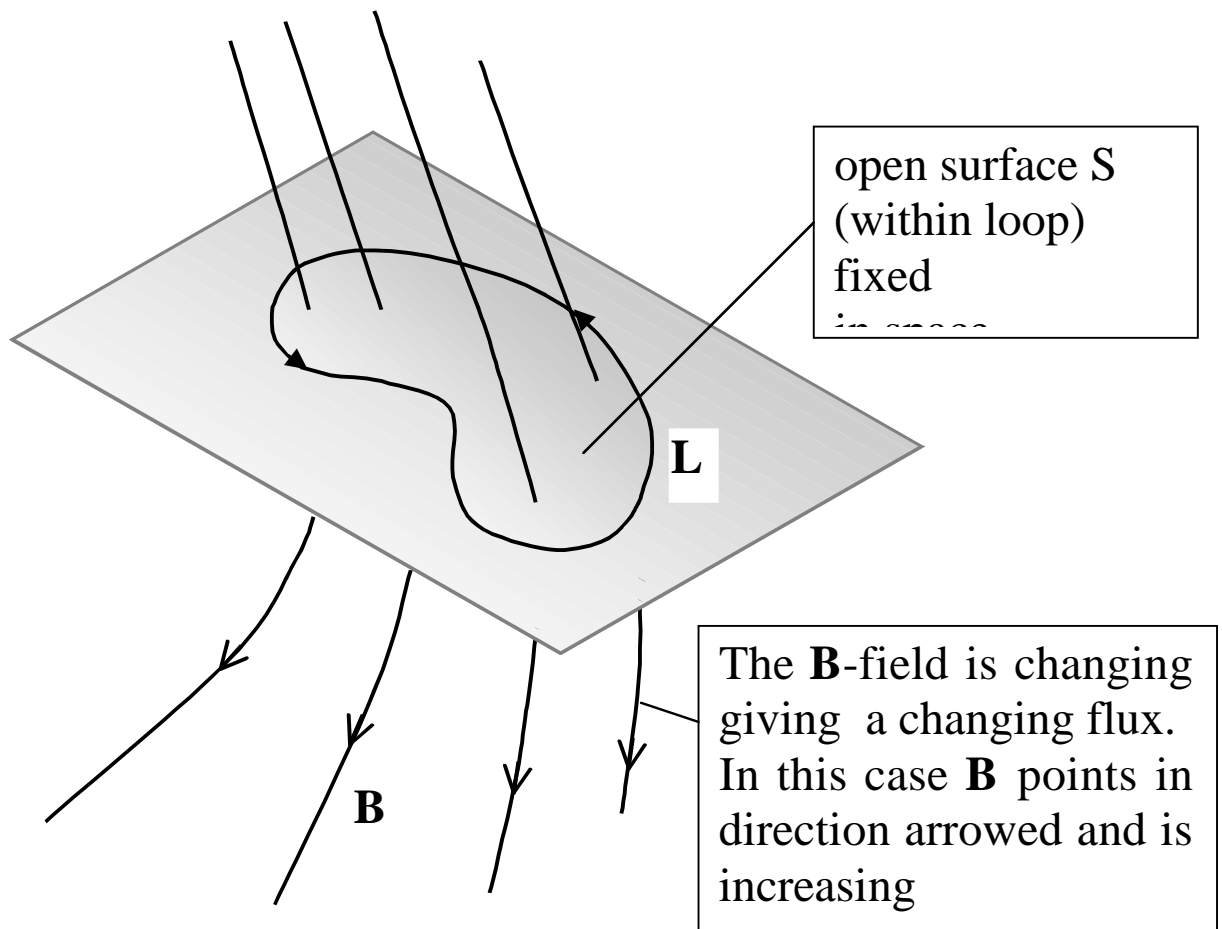
Lines of magnetic induction \mathbf{B} pass through the closed surface S .

The net outward flux ($\text{div} \mathbf{B}$) through the surface is zero.

Gauss divergence theorem states, for any ‘well-behaved’ vector field \mathbf{A} ,

$$\underbrace{\int_{\tau} (\nabla \cdot \mathbf{A}) d\tau}_{\text{integral of } \text{div} \mathbf{A} \text{ over the volume } \tau} = \underbrace{\oint_S \mathbf{A} \cdot d\mathbf{a}}_{\text{integral of flux through surface } S \text{ enclosing volume } \tau}$$

(iii) Faraday's law $\oint_L \mathbf{E} \cdot d\mathbf{l} = - \frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{a}$

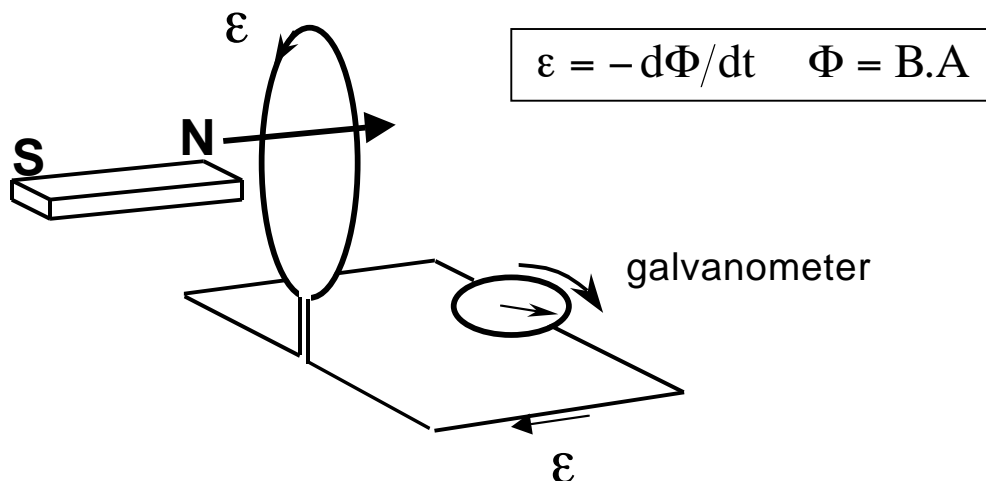


The emf induced in the loop L defined on surface S is equal to the rate of change of the magnetic flux through the surface enclosed by L.

$$\underbrace{\oint_L \mathbf{E} \cdot d\mathbf{l}}_{\text{emf in loop} \equiv \text{vector integral of electric field around L}} = - \frac{d}{dt} \underbrace{\int_S \mathbf{B} \cdot d\mathbf{a}}_{\substack{\text{rate of change of magnetic flux through surface defined by loop L} \\ \text{flux through S}}}$$

Faraday's law is easy to see (and you'll recognise it from first year!) if we take a physical (actual) loop (e.g. of wire):

Permanent magnet moved into loop of wire

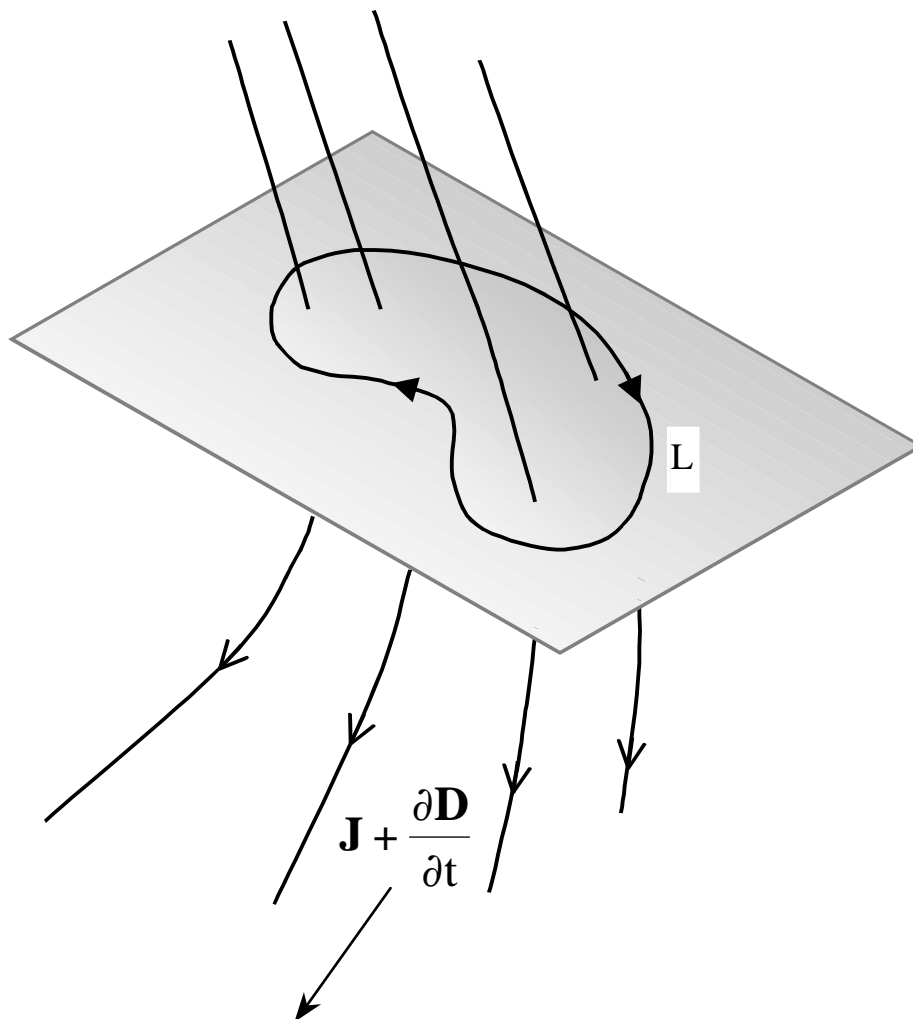


provides a changing magnetic flux (the $-\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{a}$ term) generating an emf in the wire (the $\oint_L \mathbf{E} \cdot d\mathbf{l}$ term; electric field \mathbf{E} established the emf) which deflects the galvanometer.

$$(iv) \oint_L \mathbf{H} \cdot d\mathbf{l} = I_{f,enc} + \frac{d}{dt} \int_S \mathbf{D} \cdot d\mathbf{a} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{a}$$

or, equivalently,

$$\oint_L \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int_S \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \cdot d\mathbf{a}$$



The term $\oint_L \mathbf{H} \cdot d\mathbf{l}$ is equal to the sum of two contributions linking loop L

- (i) the free current \mathbf{J} ,
- (ii) the displacement current $\frac{\partial \mathbf{D}}{\partial t}$

The arrows on the diagram above give the direction of the free current \mathbf{J} .

If \mathbf{D} is downward and increasing or upward and decreasing the displacement current $\mathbf{J}_D = \frac{\partial \mathbf{D}}{\partial t}$ is also in the downward direction