5 Relativistic Effects on Satellite Clocks

For atomic clocks in satellites, it is most convenient to consider the motions as they would be observed in the local ECI frame. Then the Sagnac effect becomes irrelevant. (The Sagnac effect on moving ground-based receivers must still be considered.) Gravitational frequency shifts and second-order Doppler shifts must be taken into account together. In this section I shall discuss in detail these two relativistic effects, using the expression for the elapsed coordinate time, Eq. (28). The term Φ_0 in Eq. (28) includes the scale correction needed in order to use clocks at rest on the earth's surface as references. The quadrupole contributes to Φ_0 in the term $-GM_E J_2/2a_1$ in Eq. (28); there it contributes a fractional rate correction of -3.76×10^{-13}. This effect must be accounted for in the GPS. Also, *V* is the earth's gravitational potential at the satellite's position. Fortunately, the earth's quadrupole potential falls off very rapidly with distance, and up until very recently its effect on satellite vehicle (SV) clock frequency has been neglected. This will be discussed in a later section; for the present I only note that the effect of earth's quadrupole potential on SV clocks is only about one part in 10^{14}, and I neglect it for the moment.

Satellite orbits. Let us assume that the satellites move along Keplerian orbits. This is a good approximation for GPS satellites, but poor if the satellites are at low altitude. This assumption yields relations with which to simplify Eq. (28). Since the quadrupole (and higher multipole) parts of the earth's potential are neglected, in Eq. (28) the potential is $V = -GM_E/r$. Then the expressions can be evaluated using what is known about the Newtonian orbital mechanics of the satellites. Denote the satellite's orbit semimajor axis by *a* and eccentricity by *e*. Then the solution of the orbital equations is as follows [13]: The distance *r* from the center of the earth to the satellite in ECI coordinates is

$$r = a(1 - e^2)/(1 + e \cos f). \qquad (29)$$

The angle *f*, called the true anomaly, is measured from perigee along the orbit to the satellite's instantaneous position. The true anomaly can be calculated in terms of another quantity *E* called the eccentric anomaly, according to the relationships

$$\cos f = \frac{\cos E - e}{1 - e\cos E},$$
$$\sin f = \sqrt{1-e^2}\frac{\sin E}{1 - e\cos E}. \quad (30)$$

Then, another way to write the radial distance *r* is

$$r = a(1 - e\cos E). \quad (31)$$

To find the eccentric anomaly *E*, one must solve the transcendental equation

$$E - e\sin E = \sqrt{\frac{GM_E}{a^3}}(t - t_p), \quad (32)$$

where t_p is the coordinate time of perigee passage.

In Newtonian mechanics, the gravitational field is a conservative field and total energy is conserved. Using the above equations for the Keplerian orbit, one can show that the total energy per unit mass of the satellite is

$$\frac{1}{2}v^2 - \frac{GM_E}{r} = -\frac{GM_E}{2a}. \quad (33)$$

If I use Eq. (33) for v^2 in Eq. (28), then I get the following expression for the elapsed coordinate time on the satellite clock:

$$\Delta t = \int_{\mathrm{path}} d\tau \left[1 + \frac{3GM_E}{2ac^2} + \frac{\Phi_0}{c^2} - \frac{2GM_E}{c^2}\left(\frac{1}{a} - \frac{1}{r}\right)\right]. \quad (34)$$

The first two constant rate correction terms in Eq. (34) have the values:

$$\frac{3GM_E}{2ac^2} + \frac{\Phi_0}{c^2} = +2.5046\times 10^{-10} - 6.9693\times 10^{-10} = -4.4647\times 10^{-10}. \quad (35)$$

The negative sign in this result means that the standard clock in orbit is beating too fast, primarily because its frequency is gravitationally blueshifted. In order for the satellite clock to appear to an observer on the geoid to beat at the chosen frequency of 10.23 MHz, the satellite clocks are adjusted lower in frequency so that the proper frequency is:

$$\left[1 - 4.4647 \times 10^{-10}\right] \times 10.23\ \mathrm{MHz} = 10.229\,999\,995\,43\ \mathrm{MHz}. \qquad (36)$$

This adjustment is accomplished on the ground before the clock is placed in orbit.

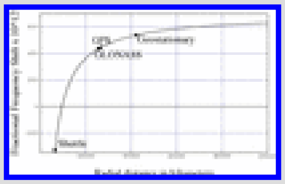

Figure 2: *Net fractional frequency shift of a clock in a circular orbit.*

Figure 2 shows the net fractional frequency offset of an atomic clock in a circular orbit, which is essentially the left side of Eq. (35) plotted as a function of orbit radius *a*, with a change of sign. Five sources of relativistic effects contribute in Figure 2. The effects are emphasized for several different orbit radii of particular interest. For a low earth orbiter such as the Space Shuttle, the velocity is so great that slowing due to time dilation is the dominant effect, while for a GPS satellite clock, the gravitational blueshift is greater. The effects cancel at $a \approx 9545\ \mathrm{km}$. The Global Navigation Satellite System Galileo, which is currently being designed under the auspices of the European Space Agency, will have orbital radii of approximately 30,000 km.

There is an interesting story about this frequency offset. At the time of launch of the NTS-2 satellite (23 June 1977), which contained the first Cesium atomic clock to be placed in orbit, it was recognized that orbiting clocks would require a relativistic correction, but there was uncertainty as to its magnitude as well as its sign. Indeed, there were some who doubted that relativistic effects were truths that would need to be incorporated [5]! A frequency synthesizer was built into the satellite clock system so that after launch, if in fact the rate of the clock in its final orbit was that predicted by general relativity, then the synthesizer could be turned on, bringing the clock to the coordinate rate necessary for operation. After the Cesium clock was turned on in NTS-2, it was operated for about 20 days to measure its clock rate before turning on the synthesizer [11]. The frequency measured during that interval was +442.5 parts in 10^{12} compared to clocks on the ground, while general relativity predicted +446.5 parts in 10^{12}. The difference was well within the accuracy capabilities of the orbiting clock. This then gave about a 1% verification of the combined second-order Doppler and gravitational frequency shift effects for a clock at 4.2 earth radii.

Additional small frequency offsets can arise from clock drift, environmental changes, and other unavoidable effects such as the inability to launch the satellite into an orbit with precisely the desired semimajor axis. The navigation message provides satellite clock frequency corrections for users so that in effect, the clock frequencies remain as close as possible to the frequency of the U.S.\ Naval Observatory's reference clock ensemble. Because of such effects, it would now be difficult to use GPS clocks to measure relativistic frequency shifts.

When GPS satellites were first deployed, the specified factory frequency offset was slightly in error because the important contribution from earth's centripetal potential (see Eq. (18)) had been inadvertently omitted at one stage of the evaluation. Although GPS managers were made aware of this error in the early 1980s, eight years passed before system specifications were changed to reflect the correct calculation [2]. As understanding of the numerous sources of error in the GPS slowly improved, it eventually made sense to incorporate the correct relativistic calculation.

The eccentricity correction. The last term in Eq. (34) may be integrated exactly by using the following expression for the rate of change of eccentric anomaly with time, which follows by differentiating Eq. (32):

$$\frac{dE}{dt} = \frac{\sqrt{GM_{\mathrm{E}}/a^3}}{1 - e\cos E}. \tag{37}$$

Also, since a relativistic correction is being computed, $ds/c \simeq dt$, so

$$\begin{aligned}
\int \left[\frac{2GM_{\mathrm{E}}}{c^2}\left(\frac{1}{r} - \frac{1}{a}\right)\right]\frac{ds}{c} &\simeq \frac{2GM_{\mathrm{E}}}{c^2}\int\left(\frac{1}{r} - \frac{1}{a}\right)dt \\
&= \frac{2GM_{\mathrm{E}}}{ac^2}\int dt\left(\frac{e\cos E}{1 - e\cos E}\right) \\
&= \frac{2\sqrt{GM_{\mathrm{E}}a}}{c^2}e\left(\sin E - \sin E_0\right) \\
&= +\frac{2\sqrt{GM_{\mathrm{E}}a}}{c^2}e\sin E + \mathrm{constant}.
\end{aligned} \tag{38}$$

The constant of integration in Eq. (38) can be dropped since this term is lumped with other clock offset effects in the Kalman filter computation of the clock correction model. The net correction for clock offset due to relativistic effects that vary in time is

$$\Delta t_{\mathrm{r}} = +4.4428 \times 10^{-10} e\sqrt{a}\sin E\ \frac{\mathrm{s}}{\sqrt{\mathrm{m}}}. \tag{39}$$

This correction must be made by the receiver; it is a correction to the coordinate time as transmitted by the satellite. For a satellite of eccentricity $e = 0.01$, the maximum size of this term is about 23 ns. The correction is needed because of a combination of effects on the satellite clock due to gravitational frequency shift and second-order Doppler shift, which vary due to orbit eccentricity.

Eq. (39) can be expressed without approximation in the alternative form

$$\Delta t_r = +\frac{2\mathbf{r} \cdot \mathbf{v}}{c^2}, \qquad (40)$$

where $\mathbf{r}$ and $\mathbf{v}$ are the position and velocity of the satellite at the instant of transmission. This may be proved using the expressions (30, 31, 32) for the Keplerian orbits of the satellites. This latter form is usually used in implementations of the receiver software.

It is not at all necessary, in a navigation satellite system, that the eccentricity correction be applied by the receiver. It appears that the clocks in the GLONASS satellite system do have this correction applied before broadcast. In fact historically, this was dictated in the GPS by the small amount of computing power available in the early GPS satellite vehicles. It would actually make more sense to incorporate this correction into the time broadcast by the satellites; then the broadcast time events would be much closer to coordinate time - that is, GPS system time. It may now be too late to reverse this decision because of the investment that many dozens of receiver manufacturers have in their products. However, it does mean that receivers are supposed to incorporate the relativity correction; therefore, if appropriate data can be obtained in raw form from a receiver one can measure this effect. Such measurements are discussed next.

Relativity in the Global Positioning System
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