9. ELECTROMAGNETIC WAVES

9.1 Classical wave equation

We saw in PHYS1231 (!) how Newton's law $\mathbf{F} = m\mathbf{a}$ can be applied to an element of **string** under tension to derive the differential equation describing waves propagating with velocity υ.

$$\frac{\partial^2 f}{\partial z^2} = \frac{1}{\upsilon^2}\frac{\partial^2 f}{\partial t^2} \qquad \textbf{classical wave equation}$$

(in the case of waves on a string $\upsilon = \sqrt{T/\mu}$; T = tension, μ = string's mass per unit length).
 (see Tipler p446-448 (wave on string) and p1012-1014 (EM wave)

9.2 Wave equation solutions

Solutions of the classical wave equation are functions with the form

$$f(z,t) \;=\; g(z \pm \upsilon t)$$

+ describes a wave travelling in $-$ve z-dir^n.

$-$ describes a wave travelling in +ve z-dir^n.

and since the wave equation is *linear* (there are no differentials raised to power 2 or higher) we can have superpositions of solutions too:

$$f(z,t) = \underbrace{g(z - \upsilon t) + h(z + \upsilon t)}$$

wave travelling toward +ve z plus wave travelling toward −ve z is also a solution of the wave equation
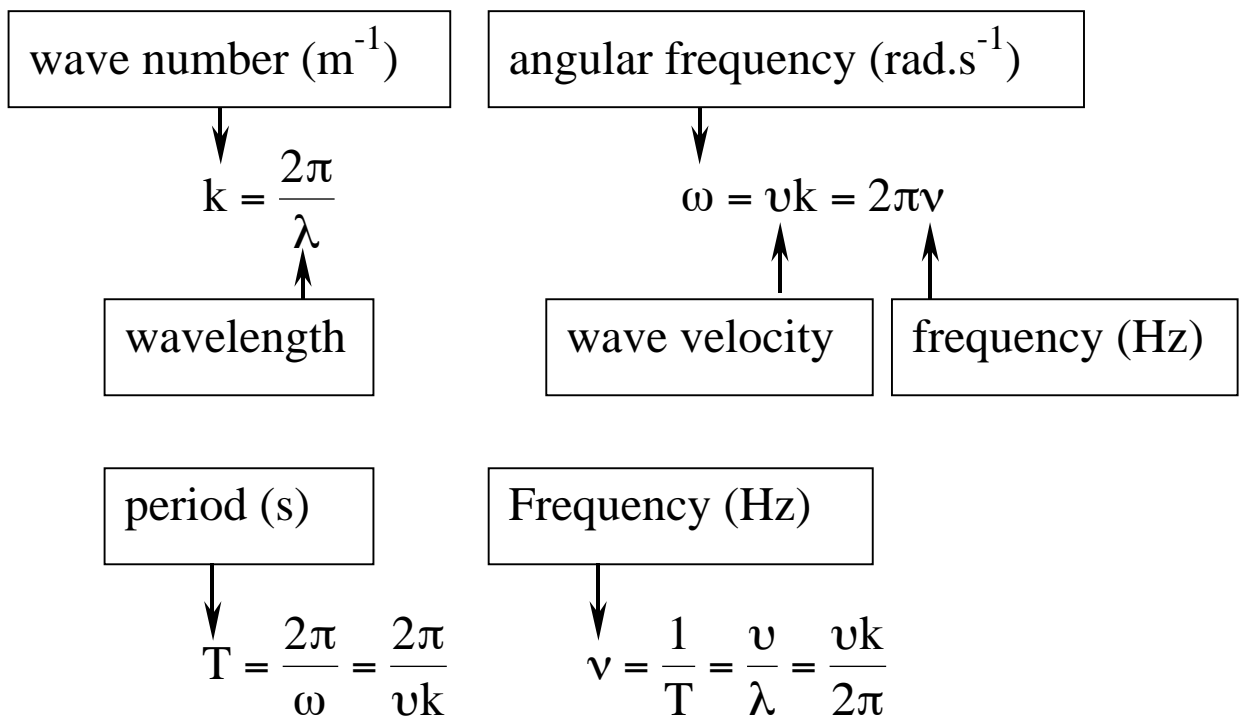
Of all the (many!) possible mathematical functions *g* that are solutions to the wave equation, harmonic (sine and cosine) functions are the most useful:

phase

$$f(z,t) = A \cos\left[k(z - \upsilon t) + \delta\right]$$

amplitude

velocity

phase constant

wave number

We can re-write $f(z,t)$ in various useful forms using

| wave number (m^{-1}) | angular frequency (rad.s^{-1}) |

$$k = \frac{2\pi}{\lambda}$$

$$\omega = \upsilon k = 2\pi\nu$$

| wavelength | wave velocity | frequency (Hz) |

| period (s) | Frequency (Hz) |

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\upsilon k}$$

$$\nu = \frac{1}{T} = \frac{\upsilon}{\lambda} = \frac{\upsilon k}{2\pi}$$

A very useful form of solution is

$$f(z,t) = A\cos(kz \pm \omega t \pm \delta)$$

9.3 Wave equation solution in complex notation

Recalling that $e^{i\theta} = \cos\theta + i\sin\theta$, we can write

$$f(z,t) = \mathrm{Re}\left[Ae^{i(kz-\omega t+\delta)}\right]$$

Re: take the real part of
this complex exponential

If we make the wave function complex by including a complex amplitude $\tilde{A} = Ae^{i\delta}$ we write:

$$\tilde{f}(z,t) = \tilde{A}e^{i(kz-\omega t)}$$

9.4 Maxwell's equations give propagating EM waves

If we take the two Maxwell eqns. (in differential form)

(iii) $\quad \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$ $\qquad$ (Faraday's law)

(iv) $\quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \dfrac{\partial \mathbf{E}}{\partial t}$ $\quad$ (Ampère's law with Maxwell's correction)

and take the curl of these two in a region of space with

- *no charge*
- *no current*

Take curl of (iii):

$$\nabla \times \left(\nabla \times \mathbf{E}\right) = \nabla\left(\nabla . \mathbf{E}\right) - \nabla^2 \mathbf{E} = \nabla \times \left(-\dfrac{\partial \mathbf{B}}{\partial t}\right)$$
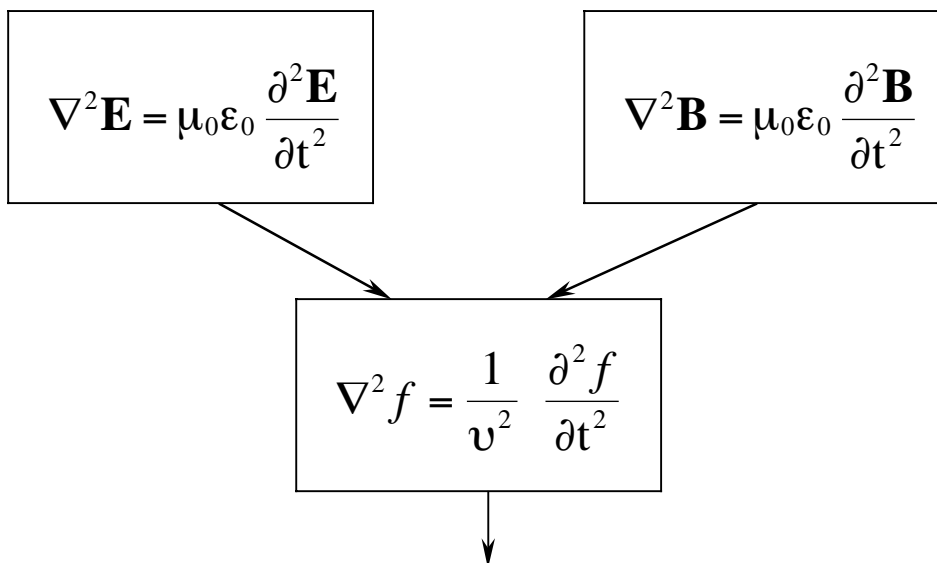
$$= -\frac{\partial}{\partial t}\left(\nabla \times \mathbf{B}\right) = -\mu_0\varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

take curl of (iv)

$$\nabla \times \left(\nabla \times \mathbf{B}\right) = \nabla\left(\nabla.\mathbf{B}\right) - \nabla^2\mathbf{B} = \nabla \times \left(\mu_0\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right)$$

$$= -\mu_0\varepsilon_0 \frac{\partial}{\partial t}\left(\nabla \times \mathbf{E}\right) = -\mu_0\varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}$$

The equations for **E** and **B**

$$\nabla^2\mathbf{E} = \mu_0\varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

$$\nabla^2\mathbf{B} = \mu_0\varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}$$

$$\nabla^2 f = \frac{1}{\upsilon^2} \frac{\partial^2 f}{\partial t^2}$$

with

$$\upsilon = \frac{1}{\sqrt{\varepsilon_0\mu_0}} = 3.00 \times 10^8 \text{ m/s}$$

These equations (obviously (!)) describe electromagnetic waves EM propagating at velocity $\upsilon = c$.

James Clerk Maxwell (1831-1879), Scottish mathematician and physicist in "*A Treatise on Electricity and Magnetism*" (1873) first showed that the (eponymous; Maxwell) equations implicitly require the existence of EM waves travelling at the speed of light.

Maxwell also knew the numerical values of μ_0 and ε_0 measured in Germany in 1856. The speed of light was also known.

History: Speed of light first investigated by Ole Römer in 1676. Römer observer the eclipse of Io, one of Jupiter's moons: Römer found the speed of light to be very large…but finite!

Armand Fizeau used the Fizeau Wheel (a rotating toothed wheel and distant mirror) and found $c = 3.15 \times 10^8$ ms^{-1} in ~1850.
(http://scienceworld.wolfram.com/physics/FizeauWheel.html and links therein is an interesting source)

9.5 Power in EM waves

Electromagnetic waves transmit *information* and *power*. We are already familiar with the energy density (per unit volume) in static magnetic and electric fields:

$$u = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)$$

and since the electric and magnetic components contribute equally (see Griffiths p 378, Example 9.2),

$$B^2 = \frac{1}{c^2}E^2 = \mu_0\varepsilon_0 E^2$$

and,

$$u = \varepsilon_0 E^2 = \varepsilon_0 E_0^2 \cos^2\left(kz - \omega t + \delta\right) \qquad\qquad *$$

> energy density in the
> electromagnetic field

The **Poynting vector S** gives the energy per unit area per second and is defined

$$S = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B})$$

and using * above,

$$S = c\varepsilon_0 E_0^2 \cos^2(kz - \omega t + \delta)\hat{\mathbf{z}} = cu\,\hat{\mathbf{z}}$$

energy density x
velocity of EM waves

> Note that EM waves also carry momentum $\wp = \frac{1}{c^2}S$ you can
>
> read about this on p381 of Griffiths)

For the sine squared and cos squared functions, averaged over a

cycle,

$$\langle \sin^2 \rangle = \langle \cos^2 \rangle = 1/2$$

$$\left\{ \frac{1}{T}\int_0^T \cos^2(kz - 2\pi t/T + \delta)dt = 1/2 \right\}$$

so that
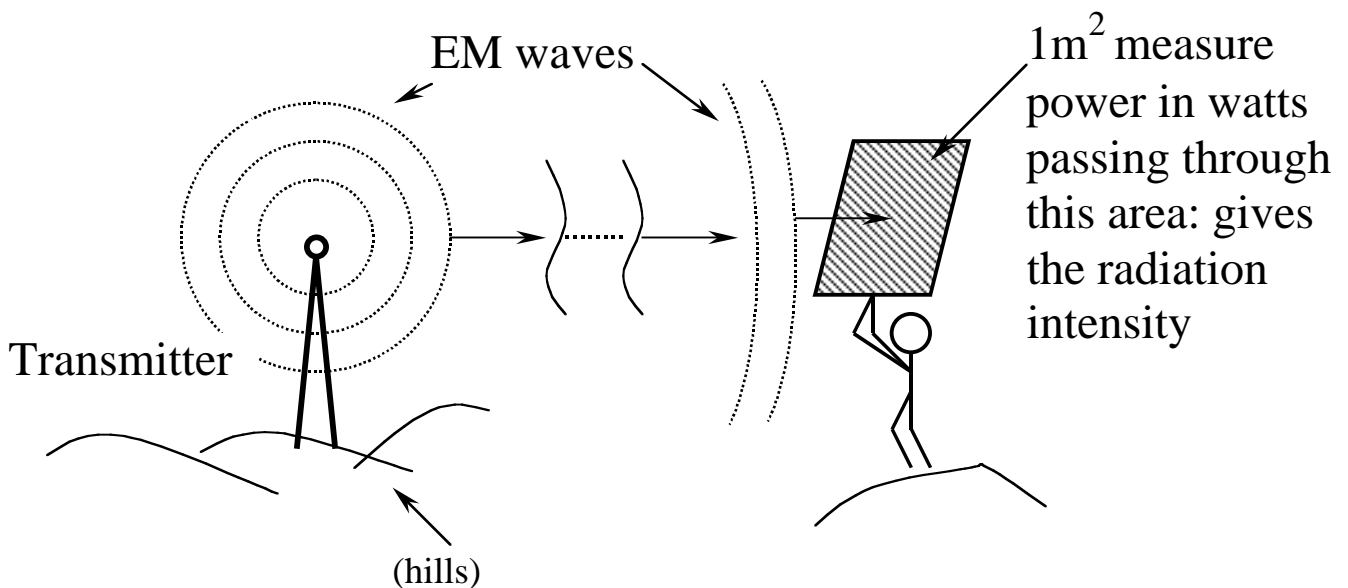$$\langle u \rangle = \frac{1}{2}\varepsilon_0 E_0^2$$

and

$$\langle \mathbf{S} \rangle = \frac{1}{2} c \varepsilon_0 E_0^2 \, \hat{\mathbf{z}}$$

The EM wave intensity is

$$I \equiv \langle \mathbf{S} \rangle = \frac{1}{2} c \varepsilon_0 E_0^2$$

This is the **average power per unit area** transmitted or transported

by the wave. This is the intensity we met in PHYS1231 when

discussing waves, in particular interference and diffraction of EM

waves.

Problems 8, 9 and 10 on Problem Sheet 6 (the last sheet!) concern

EM wave calculations.